

A theory of bundles over posets

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1 Abstract

In algebraic quantum field theory the spacetime manifold is replaced by a suitable base for its topology ordered under inclusion. We explain how certain topological invariants of the manifold can be computed in terms of the base poset. We develop a theory of connections and curvature for bundles over posets in search of a formulation of gauge theories in algebraic quantum field theory.

2 Introduction

This paper is motivated by Quantum Field Theory. Whilst the pseudo-Riemannian spacetime manifold enters directly into the formulation of Classical Field Theory, it enters into Quantum Field Theory only through a suitable base for its topology ordered under inclusion. This raises the question as to what extent the topological data of the spacetime manifold are still encoded in the partially ordered set. Positive answers have been given as far as connectivity [9] and the fundamental group go [9, 12] and we shall see in this paper that the same applies to the first locally constant cohomology group.

Perhaps the most important open question in Quantum Field Theory concerns gauge theories. There is no formulation of gauge theories which

goes beyond a perturbative framework. Whilst it is too much to hope for a single rigorous example of a gauge theory in the near future, one might hope that certain of their structural features could be axiomatized so as to be able to predict aspects of their behaviour. In particular, one might hope to say something about their superselection structure which remains a problem even for Quantum Electrodynamics.

Classical gauge theories are formulated in the language of differential geometry: there are principal bundles, associated bundles, connections and curvature, all referring to the underlying spacetime manifold. The very least change to be made to adapt the formalism to the quantum case would be to use the analogues of the structures of differential geometry over the poset derived from a base for the topology of the spacetime manifold. These analogues form the subject of this paper.

Here we now provide an outline of the paper.

In Section 3 we describe a symmetric simplicial set $\tilde{\Sigma}_*(K)$ associated with a poset K . In particular we observe that the nerve $\Sigma_*(K)$ is a subsimplicial set and that the fundamental groupoid of $\tilde{\Sigma}_*(K)$ is isomorphic to that of $\tilde{\Sigma}_*(K^\circ)$ associated with the opposite poset K° .

In Section 4 we introduce a class of bundles over posets: the *net bundles*. A net bundle \mathcal{B} over a poset K yields a fibration of symmetric simplicial sets $\pi_* : \tilde{\Sigma}_*(\mathcal{B}) \rightarrow \tilde{\Sigma}_*(K)$ with a *net structure* J : a 1-cocycle of the nerve $\Sigma_*(K)$ taking values in the groupoid of bijections between the fibres. The net structure is used in place of continuity (or differentiability) and morphisms, cross sections and connections should be compatible with the net structure. A connection, in particular, is defined as an extension of the net structure to the simplicial set $\tilde{\Sigma}_*(K)$. An important feature is that any net bundle admits one, and only one, *flat connection*: the unique 1-cocycle $\tilde{\Sigma}_*(K)$ extending the net structure J (Proposition 4.4). In accordance with these ideas, principal net bundles are defined in Section 5 by adding a suitable action of a group on the total space.

The analysis of principal net bundles relies on the observation that local representatives of connections, morphisms and cross sections are all *locally constant*. So, in Section 6, we exploit this feature to provide an equivalent description of principal net bundles and their connections in terms of the cohomology of posets as described in [10]: the category of principal net bundles of a poset K having structure group G is equivalent to the category of 1-cocycles of K taking values in G (and analogously for connections, see Theorem 6.7). Afterwards, in Section 7, we transport the notion of curvature, the relation between flatness and homotopy, the existence of non-

flat connections, and the Ambrose-Singer theorem from the cohomology of posets to principal net bundles.

In Section 8, we introduce the Čech cohomology of a poset. Then, we specialize our discussion to the poset of open, contractible subsets of a manifold. In this case, the above constructions yield the locally constant cohomology of the manifold, which, as is well known, describes the category of flat bundles (Theorem 8.2 and Proposition 8.4).

We conclude the paper with an appendix recalling briefly the results of [10].

3 Homotopy of posets

In this section we analyze the simplicial sets associated with a poset. We start by introducing symmetric simplicial sets and defining their fundamental groupoid. Afterwards, we consider symmetric simplicial sets associated with categories and establish a relation between the fundamental groupoid of a category and that of the corresponding opposite category. Then we specialize to posets. We conclude with some remarks on the topology of partially ordered sets. References for this section are [7, 10].

3.1 Simplicial Sets

Our investigation of the relation between invariants of a topological space and those of a suitable base for its topology ordered under inclusion makes substantial use of simplicial sets. A simplicial set is a contravariant functor from the simplicial category Δ^+ to the category of sets. Δ^+ is a subcategory of the category of sets having as objects $n := \{0, 1, \dots, n-1\}$, $n \in \mathbb{N}$ and as mappings the order preserving mappings. A simplicial set has a well known description in terms of generators, the face and degeneracy maps, and relations. We use the standard notation ∂_i and σ_j for the face and degeneracy maps, and denote the compositions $\partial_i \partial_j$, $\sigma_i \sigma_j$, respectively, by ∂_{ij} , σ_{ij} . A path in a simplicial set is an expression of the form $p := b_n * b_{n-1} * \dots * b_1$ where the b_i are 1-simplices and $\partial_0 b_i = \partial_1 b_{i+1}$ for $i = 1, 2, \dots, n-1$. We set $\partial_1 p := \partial_1 b_1$ and $\partial_0 p := \partial_0 b_n$. Concatenation gives us an obvious associative composition law for paths and in this way we get a category without units.

Homotopy provides us with an equivalence relation on this structure. This is the equivalence relation generated by saying that two paths of the form $p = b_n * b_{n-1} * \dots * b_i * \partial_1 c * b_{i-1} * \dots * b_1$ and $q = b_n * b_{n-1} * \dots * b_i * \partial_0 c *$

$\partial_2 c * b_{i-1} * \cdots * b_1$, where c is a 2-simplex, are equivalent, $p \sim q$. Quotienting by this equivalence relation yields the homotopy category of the simplicial set.

We shall mainly use symmetric simplicial sets. These are contravariant functors from Δ^s to the category of sets, where Δ^s is the full subcategory of the category of sets with the same objects as Δ^+ . A symmetric simplicial set also has a description in terms of generators and relations, where the generators now include the permutations of adjacent vertices, denoted τ_i . In a symmetric simplicial set we define the reverse of a 1-simplex b to be the 1-simplex $\bar{b} := \tau_0 b$ and the reverse of a path $p = b_n * b_{n-1} * \cdots * b_1$ is the path $\bar{p} := \tau_0 b_1 * \tau_0 b_2 * \cdots * \tau_0 b_n$. The reverse acts as an inverse after taking equivalence classes so the homotopy category becomes a homotopy groupoid.

We shall be concerned here with three different simplicial sets that can be associated with a poset and we give their definitions not just for a poset but for an arbitrary category \mathcal{C} . The first denoted $\Sigma_*(\mathcal{C})$ is just the usual nerve of the category. Thus the 0-simplices are just the objects of \mathcal{C} , the 1-simplices are the arrows of \mathcal{C} and a 2-simplex c is made up of its three faces which are arrows satisfying $\partial_0 c \partial_2 c = \partial_1 c$. The explicit form of higher simplices will not be needed in this paper. The homotopy category of $\Sigma_*(\mathcal{C})$ is canonically isomorphic to \mathcal{C} itself.

The second simplicial set is just the nerve $\Sigma_*(\hat{\mathcal{C}})$ of the category of fractions $\hat{\mathcal{C}}$ of \mathcal{C} [2]. The proof of the result in this section requires some knowledge of how $\hat{\mathcal{C}}$ is constructed. Consider \mathcal{C} and the opposite category \mathcal{C}° . These categories have the same set of objects so we may consider paths $p := b_n * b_{n-1} * \cdots * b_1$ where $\partial_0 b_i = \partial_1 b_{i+1}$, $i = 1, 2, \dots, n-1$ as before but now the b_i can be taken at liberty to be arrows of \mathcal{C} or \mathcal{C}° . We now take the equivalence relation generated by homotopy within adjacent arrows of \mathcal{C} , homotopy within adjacent arrows of \mathcal{C}° and two further relations $p * b * b^{-1} * q \sim p * q$ where b^{-1} denotes the arrow corresponding to b in the corresponding opposite category and where p and q are not both the empty path. Finally $b * b^{-1} \sim 1_{\partial_0 b}$. Quotienting by this equivalence relation yields the category of fractions $\hat{\mathcal{C}}$. The arrows of $\hat{\mathcal{C}}$ can be written in normal form. The units of $\hat{\mathcal{C}}$ on their own are in normal form. The other terms in normal form involve alternate compositions of arrows of \mathcal{C} and of \mathcal{C}° but no units nor compositions of an arrow and its inverse. Every arrow of $\hat{\mathcal{C}}$ is invertible so that $\Sigma_*(\hat{\mathcal{C}})$ is in a natural way a symmetric simplicial set.

The third simplicial set $\tilde{\Sigma}_*(\mathcal{C})$ is also symmetric and is constructed as follows. Consider the poset P_n of non-void subsets of $\{0, 1, \dots, n-1\}$ ordered

under inclusion. Any mapping f from $\{0, 1, \dots, m-1\}$ to $\{0, 1, \dots, n-1\}$ induces an order preserving mapping from P_m to P_n . Regarding the P_n as categories, we have realized Δ^s as a subcategory of the category of categories. We then get a symmetric simplicial set where an n -simplex of $\tilde{\Sigma}_*(\mathcal{C})$ is a functor from P_n to \mathcal{C} . A 1-simplex of $\tilde{\Sigma}_*(\mathcal{C})$ is a pair (b_0, b_1) of arrows of \mathcal{C} with $\partial_0 b_0 = \partial_0 b_1$ and $\partial_0(b_0, b_1) = \partial_1 b_0$, $\partial_1(b_0, b_1) = \partial_1 b_1$. A 2-simplex of $\tilde{\Sigma}_*(\mathcal{C})$ is a set

$$(c_0, c_1, c_2; c_{01}, c_{02}, c_{10}, c_{12}, c_{20}, c_{21})$$

of nine arrows of \mathcal{C} with

$$c_0 c_{00} = c_1 c_{10}, \quad c_0 c_{01} = c_2 c_{20}, \quad c_1 c_{11} = c_2 c_{20}.$$

The faces of this 2-simplex c are given by $\partial_0 c = (c_{20}, c_{10})$, $\partial_1 c = (c_{21}, c_{01})$, $\partial_2 c = (c_{12}, c_{02})$. An explicit description of the higher simplices will not be needed.

Theorem 3.1. *The homotopy groupoids of $\Sigma_*(\hat{\mathcal{C}})$ and $\tilde{\Sigma}_*(\mathcal{C})$ are isomorphic.*

Proof. Given a path $p := b_0 * b_1 * \dots * b_n$ in $\tilde{\Sigma}_1(\mathcal{C})$ we define a map ϕ into paths in $\Sigma_1(\hat{\mathcal{C}})$, setting $\phi(p) := \phi(b_0)\phi(b_1) \dots \phi(b_n)$, where for $b := (b_0, b_1)$, $\phi(b) := b_0^{-1} b_1 \in \Sigma_1(\hat{\mathcal{C}})$. If $p \sim p'$ then using the barycentric decomposition inherent in a 2-simplex c of $\tilde{\Sigma}_*(\mathcal{C})$, we see that

$$c_{00}^{-1} c_{01} c_{20}^{-1} c_{21} = c_{10}^{-1} c_{11}.$$

Hence $\phi(p) \sim \phi(p')$. Conversely, given a path $\hat{p} \in \Sigma_1(\hat{\mathcal{C}})$,

$$\hat{p} = \hat{b}_0 * \hat{b}_1 * \dots * \hat{b}_n,$$

each \hat{b}_i being an arrow in $\hat{\mathcal{C}}$, we set $\psi(\hat{p}) = \psi(\hat{b}_0) * \psi(\hat{b}_1) * \dots * \psi(\hat{b}_n)$ where $\psi(\hat{b}_i)$ is defined using the normal form of \hat{b}_i .

If the normal form of \hat{b} is $b_0 b_1 \dots b_m$ then $\psi(\hat{b}) = \psi(b_0) * \psi(b_1) * \dots * \psi(b_m)$ where $\psi(b) := (\sigma_0 \partial_0 b, b)$ for b an arrow of \mathcal{C} and $\psi(b^{-1}) := (b, \sigma_0 \partial_0 b)$ when b^{-1} is an arrow of \mathcal{C}° . For a unit we have $\psi(\sigma_0 a) = (\sigma_0 a, \sigma_0 a)$. We claim that $\hat{p} \sim \hat{p}'$ implies $\psi(\hat{p}) \sim \psi(\hat{p}')$. It suffices to show that $\psi(\hat{p}) \sim \psi(\hat{b}_0 \hat{b}_1 \dots \hat{b}_n)$ or that $\psi(b_0) * \dots * \psi(b_k) \sim \psi(b_0 b_1 \dots b_k)$. Thus the problem has been reduced to showing that the homotopy class of the left hand side does not change as $b_0 b_1 \dots b_k$ is put into normal form. This can be done using the following moves: composing two adjacent arrows of \mathcal{C} , composing two adjacent arrows of \mathcal{C}° , removing an arrow and its inverse if they are adjacent unless they are

the only arrows in which case they are to be replaced by the identity. In the first case it is enough to note that

$$(\sigma_0 \partial_0 b, b) * (\sigma_0 \partial_0 b', b') \sim (\sigma_0 \partial_0 b, bb')$$

as follows by taking a 2-simplex with $c_0 = c_1 = \sigma_0 \partial_0 b$ and $c_3 = b$. The second case follows by analogy. For the third case we need to know that

$$(b, \sigma_0 \partial_0 b) * (\sigma_0 \partial_0 b, b) \sim (\sigma_0 \partial_1 b, \sigma_0 \partial_1 b)$$

as follows by taking a 2-simplex c with $c_0 = c_2 = \sigma_0 \partial_0 b$ and $c_1 = b$.

To see that $\psi \circ \phi$ induces the identity on homotopy classes, it is enough to note that, choosing a 2-simplex c with $c_0 = c_1 = c_2 = \sigma_0 \partial_0 b$, $(b, \sigma_0 \partial_0 b) * (\sigma_0 \partial_0 b', b')$ is homotopic to (b, b') when $\partial_0 b = \partial_0 b'$. This same observation suffices to show that $\phi \circ \psi$ induces the identity on homotopy classes since we may suppose that \hat{p} is in normal form. Hence $\tilde{\Sigma}_*(\mathcal{C})$ and $\Sigma_*(\hat{\mathcal{C}})$ have isomorphic homotopy groupoids. \square

In view of this result we will denote the homotopy groupoid of these symmetric simplicial sets by $\pi_1(\mathcal{C})$. Since $\pi_1(\mathcal{C})$ and $\pi_1(\mathcal{C}^\circ)$ can both be calculated using $\Sigma_*(\hat{\mathcal{C}})$, $\pi_1(\mathcal{C})$ and $\pi_1(\mathcal{C}^\circ)$ are isomorphic despite the fact that the categories can apparently have very little to do with each other.

3.2 Simplicial sets of a poset

In the present paper we will work with the simplicial set $\tilde{\Sigma}_*(K)$ associated with a poset K showing that it is equivalent to that used in [10]. This will allow us to connect the cohomology of posets, developed in that paper, with bundles over posets.

To begin with, we observe that a poset is a category with at most one arrow between any two objects. This implies that any covariant functor between the categories associated with two posets induces a unique order preserving map between the posets: the mapping between the corresponding set of objects. Conversely, any order preserving map between two posets induces a functor between the corresponding categories.

Now, according to the previous observation, any n -simplex x of the symmetric simplicial set $\tilde{\Sigma}_*(K)$, defined in the previous section, can be equivalently defined as an order preserving map $f : P_n \rightarrow K$. A detailed description of $\tilde{\Sigma}_*(K)$ in these terms has been given in [10]. Here we just observe that a 0-simplex is just an element of the poset. For $n \geq 1$, an

n -simplex x is formed by $n + 1$ $(n - 1)$ -simplices $\partial_0 x, \dots, \partial_n x$, and by a 0-simplex $|x|$ called the *support* of x such that $|\partial_0 x|, \dots, |\partial_n x| \leq |x|$. The nerve $\Sigma_*(K)$ turns out to be a subsimplicial set of $\tilde{\Sigma}_*(K)$. To see this, it is enough to define $f_0(a) := a$ on 0-simplices and, inductively, $|f_n(x)| := \partial_{01\dots(n-1)} x$ and $\partial_i f_n(x) := f_{n-1}(\partial_i x)$. So we obtain a simplicial map $f_* : \Sigma_*(K) \rightarrow \tilde{\Sigma}_*(K)$ ¹. We sometimes adopt the following notation: $(o; a, \tilde{a})$ is the 1-simplex of $\tilde{\Sigma}_1(K)$ whose support is o and whose 0- and 1-face are, respectively, a and \tilde{a} ; (a, \tilde{a}) is the 1-simplex of the nerve $\Sigma_1(K)$ whose 0- and 1-face are, respectively, a and \tilde{a} .

In the present paper we will consider pathwise connected posets K . This amounts to saying that the simplicial set $\tilde{\Sigma}_*(K)$ is pathwise connected. The homotopy groupoid $\pi_1(K)$ of K is defined as $\pi_1(\tilde{\Sigma}_*(K))$. Correspondingly, when $\pi_1(K)$ is trivial we will say that K is *simply connected*. Now, a poset K is *upward* directed whenever for any pair $o, \hat{o} \in K$ there is \tilde{o} such that $o, \hat{o} \leq \tilde{o}$. It is *downward* directed if the dual poset K° is upward directed. When K is upward directed, then $\tilde{\Sigma}_*(K)$ admits a contracting homotopy. So in this case K is simply connected. However, since $\pi_1(\tilde{\Sigma}_*(K))$ is isomorphic to $\pi_1(\tilde{\Sigma}_*(K^\circ))$, K is simply connected whenever K is downward directed too. The link between the first homotopy group of a poset and the corresponding topological notion can be achieved as follows. Let M be an arcwise connected manifold, consider a basis for the topology of M whose elements are connected and simply connected, open subsets of M . Denote the poset formed by ordering this basis under *inclusion* by K . Then K is pathwise connected and $\pi_1(K)$ turns out to be isomorphic to the fundamental groupoid $\pi_1(M)$ of M [12, Theorem 2.18].

3.3 Posets as topological spaces

Let K be a poset which we will equip with a topology defined by taking $V_a := \{a' \in K : a' \leq a\}$ as a base of neighbourhoods for the topology. This topology corresponds to the Alexandroff topology on K° . The reason for choosing this convention is that the map $a \mapsto V_a$ is an order isomorphism so that any poset is a base for a topology ordered under inclusion. We denote K equipped with this topology by τK . It is easy to verify that a mapping f of posets is order preserving if and only if the corresponding mapping τf of topological spaces is continuous. Thus the category \mathcal{K} of posets may be regarded as a full subcategory of the category of topological spaces.

¹In [10] $\Sigma_*(K)$ was called the inflationary structure of $\tilde{\Sigma}_*(K)$ and denoted by $\Sigma^{\text{inf}}(K)$. This part of $\tilde{\Sigma}_*(K)$ encodes the order relation of K , explaining the terminology.

Clearly if we just know that K is a base for a topology then this cannot yield more information than can be got by supposing that τK is the topological space in question². Therefore it is worth recalling the principal features of the topological space. τK is a T_0 -space but not Hausdorff unless the ordering is trivial. The components coincide with the arcwise connected components and the associated Hausdorff space is the space of components with the discrete topology. The open set V_a is the smallest open set containing a . It has a contracting homotopy and is hence arcwise connected and simply connected. It therefore follows from [9] that the path connected components of K are in 1–1 correspondence with the arcwise connected components of τK and that $\pi_1(K)$ is isomorphic to $\pi_1(\tau K)$. Finally, we will call the open covering \mathcal{V}_0 of K defined by $\mathcal{V}_0 := \{V_a : a \in K\}$ the *fundamental covering*. Note that if \mathcal{V} is any other open covering of K , then any V_a is contained in some element of \mathcal{V} . In the following we denote $V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n}$ by $V_{a_1 a_2 \cdots a_n}$. A function f from a poset K to a set X is *locally constant* whenever $f(o) = f(\tilde{o})$ for $o \leq \tilde{o}$, or, equivalently, if $f(\partial_1 b) = f(\partial_0 b)$ for any 1-simplex b of the nerve. Another equivalent description is the following: if X is endowed with the trivial order, then $f : \tau K \rightarrow \tau X$ is continuous.

4 Bundles over posets

In this section we deal with bundles over posets without any particular structure. We discuss morphisms between bundles, connections and flat connections. We finally study the local properties, like local triviality and the existence of local cross sections. Throughout this paper the poset K is assumed to be pathwise connected.

4.1 Net bundles

We introduce the notion of a net bundle over a poset. As we shall see, a net bundle is a fibration of symmetric simplicial sets associated with posets (see Section 3.2), equipped with a cocycle of the nerve of the poset base, with values in the fibres. This cocycle replaces continuity (or differentiability) in the theory of bundles over manifolds.

Definition 4.1. A *net bundle* over a poset K is formed by a set B , a surjective map $\pi : B \rightarrow K$, and a collection J of bijective mappings

$$J_b : \pi^{-1}(\partial_1 b) \rightarrow \pi^{-1}(\partial_0 b), \quad b \in \Sigma_1(K),$$

² Knowing that the space in question is Hausdorff could yield more information.

satisfying the following relations:

- (i) $J_{\partial_0 c} J_{\partial_2 c} = J_{\partial_1 c}$ for any $c \in \Sigma_2(K)$,
- (ii) $J_{\sigma_0 a} = \text{id}_{\pi^{-1}(a)}$, for any $a \in \Sigma_0(K)$.

The symbol \mathcal{B} will denote the net bundle whose data is (B, π, J, K) .

As usual B , K and π are called, respectively, the *total space*, the *base space* and the *projection*. The subset $B_o := \pi^{-1}(o) \subset B$ is called the *fibre* over o . The collection J is called the *net structure* of the bundle. Note, in fact that the correspondence $K \ni o \rightarrow B_o \subseteq B$ with J is a net: given $o \leq \tilde{o}$ then $\text{ad}_{J_{(\tilde{o}, o)}} : B_o \rightarrow B_{\tilde{o}}$, where ad denotes the adjoint action, and (\tilde{o}, o) the 1-simplex of the nerve having 0- and 1-face respectively \tilde{o} and o . In the following symbol $\hat{\mathcal{B}}$ will indicate the net bundle $(\hat{B}, \hat{\pi}, \hat{J}, K)$.

Since K is pathwise connected and the net structure is a bijection, all the fibres of a net bundle are isomorphic. Note, however that we can weaken the above definition assuming the net structure to be just injective. Clearly, in this case the fibres are, in general, not isomorphic. We refer to these bundles as *quasinet bundles*. In the present paper we will not deal with quasinet bundles. However they will play a role in K-theory [11].

The total space B of a net bundle \mathcal{B} seems to have no structure. But this is not the case since there is an associated fibration of simplicial sets. First of all, note that given $\psi, \phi \in B$ and writing

$$\psi \leq_J \phi \iff \pi(\psi) \leq \pi(\phi) \text{ and } J_{\pi(\phi), \pi(\psi)} \psi = \phi$$

yields an order relation on B . So, the total space is a poset too with the fibres B_o having the discrete ordering. This, in turns, implies that \mathcal{B} is a fibration of simplicial sets. Consider the symmetric simplicial set $\tilde{\Sigma}_*(B)$. Set $\pi_0(\phi) := \pi(\phi)$ for any $\phi \in \tilde{\Sigma}_0(B)$, and by induction, for $n \geq 1$, define

$$\pi_n(x) := (\pi_0(|x|), \pi_{n-1}(\partial_0 x), \dots, \pi_{n-1}(\partial_n x)), \quad x \in \tilde{\Sigma}_n(B).$$

Then, we have a surjective simplicial map $\pi_* : \tilde{\Sigma}_*(B) \rightarrow \tilde{\Sigma}_*(K)$. This map is symmetric, i.e., $\pi_n \tau_i = \tau_i \pi_n$, and preserves the nerves, i.e., $\pi_* : \Sigma_*(B) \rightarrow \Sigma_*(K)$. Now, the *fibre* over a 0-simplex a is the simplicial set $\pi_*^{-1}(a)$ defined by $\pi_0^{-1}(a)$ and $\pi_{k+1}^{-1}(\sigma_{k \dots 10} a)$ for $k \geq 0$. So we have a fibration of symmetric simplicial sets whose 0-fibres are, according to our definition, isomorphic. We observe that this is not a Kan fibration since the simplicial sets involved, in general, do not fulfill the extension condition [10].

We now provide a first example of a net bundle, the *product net bundle*, introducing some notation useful in what follows. Nontrivial examples, will be given in Section 7.1. Let K be a poset, and let X be a space. Consider the Cartesian product $K \times X$ and define $\pi(o, x) := o$. Clearly, $\pi : K \times X \rightarrow K$ is a surjective map, and $\pi^{-1}(o) \simeq o \times X$. Let

$$j_b(\partial_1 b, x) := (\partial_0 b, x), \quad (b, x) \in \Sigma_1(K) \times X.$$

Clearly, $j_b : \pi^{-1}(\partial_1 b) \rightarrow \pi^{-1}(\partial_0 b)$ is a bijective map.

We now introduce morphisms between net bundles. Since in the present paper we will not need to compare net bundles over different posets, we will consider only morphisms leaving the base space invariant. The general definition can be easily obtained mimicking that for bundles over manifolds.

Definition 4.2. *Let \mathcal{B} and $\hat{\mathcal{B}}$ be net bundles over the poset K . A **net bundle morphism** f from \mathcal{B} into $\hat{\mathcal{B}}$ is a mapping of total spaces $f : B \rightarrow \hat{B}$ preserving the fibres and commuting with the net structure, namely*

$$(i) \quad \hat{\pi} f = \pi;$$

$$(ii) \quad \hat{J} f = f J.$$

*The bundles \mathcal{B} and $\hat{\mathcal{B}}$ are said to be **isomorphic** whenever $f : B \rightarrow \hat{B}$ is a bijection. A bundle isomorphic to a product bundle is said to be **trivial**.*

We conclude this section by defining the restriction of a net bundle used later to define local triviality. Let \mathcal{B} be a net bundle over K . Given an open and pathwise connected subset U of K , define $B|_U := \pi^{-1}(U)$, and $\pi|_U(\phi) := \pi(\phi)$ for $\phi \in \mathcal{B}|_U$. Moreover set $J|_{U_b} := J_b$ for any $b \in \Sigma_1(U)$. Then, $\pi_U : B|_U \rightarrow U$ is a surjective map and $J|_U$ a net structure. We call this bundle the *restriction* of \mathcal{B} to U and denote it by $\mathcal{B}|_U$.

4.2 Connections

We introduce the notion of a connection on a net bundle, and related notions like parallel transport and flat connections.

Since the net structure J of a net bundle \mathcal{B} over K is a bijection between the fibres it admits an extension from the nerve $\Sigma_1(K)$ to the simplicial set $\tilde{\Sigma}_1(K)$. In fact, let

$$Z(b) := J_{(|b|, \partial_0 b)}^{-1} J_{(|b|, \partial_1 b)}, \quad b \in \tilde{\Sigma}_1(K), \quad (1)$$

where $(|b|, \partial_i b)$ is the 1-simplex of the nerve with 0-face $|b|$ and 1-face $\partial_i b$, for $i = 0, 1$. This is well posed since the support of a 1-simplex of $\tilde{\Sigma}_1(K)$ is greater than its faces (see Section 3.2). So we have a bijection $Z(b) : B_{\partial_1 b} \rightarrow B_{\partial_0 b}$ for any $b \in \tilde{\Sigma}_1(K)$. Using the defining properties of the net structure we have

$$\begin{aligned} Z(\partial_0 c) Z(\partial_2 c) &= J_{(|\partial_0 c|, \partial_{00} c)}^{-1} J_{(|\partial_0 c|, \partial_{10} c)} J_{(|\partial_2 c|, \partial_{02} c)}^{-1} J_{(|\partial_2 c|, \partial_{12} c)} \\ &= J_{(|c|, \partial_{00} c)}^{-1} J_{(|c|, \partial_{10} c)} J_{(|c|, \partial_{02} c)}^{-1} J_{(|c|, \partial_{12} c)} \\ &= J_{(|c|, \partial_{01} c)}^{-1} J_{(|c|, \partial_{10} c)} J_{(|c|, \partial_{10} c)}^{-1} J_{(|c|, \partial_{11} c)} \\ &= J_{(|c|, \partial_{01} c)}^{-1} J_{(|c|, \partial_{11} c)} \\ &= Z(\partial_1 c), \end{aligned}$$

for any $c \in \tilde{\Sigma}_2(K)$, where the commutation relations of the faces, $\partial_{ij} = \partial_{j-1, i}$, if $i < j$, have been used (recall that ∂_{ij} stands for $\partial_i \partial_j$).

After this observation we are in a position to define connections.

Definition 4.3. A **connection** on a net bundle \mathcal{B} is a field U associating a bijective mapping $U(b) : B_{\partial_1 b} \rightarrow B_{\partial_0 b}$ to any 1-simplex b of $\tilde{\Sigma}_1(K)$, and such that

- (i) $U(\bar{b}) = U(b)^{-1}$, for $b \in \tilde{\Sigma}_1(K)$;
- (ii) $U(b) = J_{\partial_1 b, \partial_0 b}$ for any $b \in \Sigma_1(K)$.

A connection U is said to be **flat** whenever

$$U(\partial_0 c) U(\partial_2 c) = U(\partial_1 c), \quad c \in \tilde{\Sigma}_2(K).$$

We denote the set of connections of a net bundle \mathcal{B} by $\mathcal{U}(K, \mathcal{B})$.

According to this definition a connection is any extension of the net structure to the simplicial set $\tilde{\Sigma}_1(K)$; the extension Z defined by (2) is a flat connection of the net bundle. This connection is characteristic of the net bundle, as the following proposition shows.

Proposition 4.4. Z is the unique flat connection of \mathcal{B} .

Proof. Given a 1-simplex $b \in \tilde{\Sigma}_1(K)$ consider the 2-simplex $c \in \tilde{\Sigma}_2(K)$ defined by

$$|c| = |b|, \quad \partial_2 c = (|b|, |b|, \partial_1 b), \quad \partial_0 c = (|b|, \partial_0 b, |b|), \quad \partial_1 c = b.$$

Observe that $\partial_2 c$ and the reverse of $\partial_0 c$ are 1-simplices of the nerve. If U is a flat connection, then $U(b) = U(\partial_0 c) U(\partial_2 c) = J_{(|b|, \partial_0 b)}^{-1} J_{(|b|, \partial_1 b)} = Z(b)$. \square

This result is one of the main differences from the theory of bundles over manifolds. As a direct consequence, we shall see in Section 7.1 that principal net bundles over a simply connected poset are trivial. Concerning non-flat connections, we point out that, except in trivial situations, the set of non-flat connections of a principal net bundle is never empty (see Section 7.1).

Finally given a path p of the form $p = b_n * \cdots * b_1$, the *parallel transport* along p , induced by a connection U , is the mapping $U(p) : B_{\partial_1 p} \rightarrow B_{\partial_0 p}$ defined by

$$U(p) := U(b_n) \cdots U(b_2) U(b_1) . \quad (2)$$

The defining properties of a connection imply $U(\bar{p}) = U(p)^{-1}$ for any path p , and $U(\sigma_0 a) = \text{id}_{B_a}$ for any 0-simplex a .

4.3 Local triviality

We aim to show that any net bundle \mathcal{B} is *locally trivial*. Thus there is a set X and an open covering \mathcal{V} of the poset K such that, for any $X \in \mathcal{V}$, the restriction $\mathcal{B}|_X$ is equivalent to the product net bundle $V \times X$. In particular we will show that this holds for the fundamental covering \mathcal{V}_0 . This suffices for our purposes. In fact if a net bundle can be trivialized on a covering \mathcal{V} , then it can be trivialized on the fundamental covering \mathcal{V}_0 , too because $\mathcal{V}_0 \subseteq \mathcal{V}$.

Proposition 4.5. *Any net bundle \mathcal{B} over K can be trivialized on the fundamental covering \mathcal{V}_0 of K : thus there is a set X such that the restriction $\mathcal{B}|_{V_a}$ is isomorphic to the product bundle $V_a \times X$, for any $a \in K$.*

Proof. Fix $o \in K$ and define $X := \pi^{-1}(o)$. Using the flat connection Z of \mathcal{B} and pathwise connectedness, we can find a bijection $F_a : X \rightarrow \pi^{-1}(a)$ for any 0-simplex a (clearly F_a is not uniquely determined). Now recalling the definitions of restriction and product bundle we must show that there is a family of bijective mappings $\theta_a : V_a \times X \rightarrow \pi^{-1}(V_a)$, with $a \in K$, such that $\pi \theta_a = pr_1$ and $J_b \theta_a = \theta_a j_b$ for any $b \in \Sigma_1(V_a)$. To this end define

$$\theta_a(o, v) := J_{(o, a)} F_a(v), \quad (o, v) \in V_a \times X.$$

Clearly $\theta_a : V_a \times X \rightarrow \pi^{-1}(V_a)$, being the composition of bijective maps, is bijective, and $\pi \theta_a = pr_1$. Moreover, If $b \in \Sigma_1(V_a)$, then

$$J_b \theta_a(\partial_1 b, v) = J_b J_{(\partial_1 b, a)} F_a(v) = J_{(\partial_0 b, a)} F_a(v) = \theta_a j_b(\partial_1 b, v),$$

and this completes the proof. \square

The set X is called the *standard fibre* of \mathcal{B} . A family of mappings $\theta := \{\theta_a\}$, with $a \in K$, trivializing the net bundle \mathcal{B} on the fundamental covering will be called a *local trivialization* of \mathcal{B} .

We now deal with local sections of net bundles.

Definition 4.6. A *local section* of a net bundle \mathcal{B} is a map $\sigma : V \rightarrow B$, where V is an open of K , such that $\pi \sigma = \text{id}_V$, and

$$J_b(\sigma(\partial_1 b)) = \sigma(\partial_0 b), \quad b \in \Sigma_1(V).$$

If $V = K$, then σ is said to be a *global section*.

Consider a local section $\sigma : V \rightarrow B$. Let a be 0-simplex with $a \in V$, and recall that $V_a \subseteq V$. Given a local trivialization θ , define

$$\theta_a^{-1}(\sigma(o)) := (o, s_a(o)), \quad o \in V_a. \quad (3)$$

We call s_a a *local representative* of σ . An important property of net bundles, is that cross sections are *locally constant* (see Subsection 3.3). In fact, $(o, s_a(o)) = \theta_a^{-1}\sigma(o) = \theta_a^{-1}J_{(o,a)}\sigma(a) = J_{o,a}\theta_a^{-1}\sigma(a) = (o, s_a(a))$, for any $o \in V_a$. A second property is that any net bundle has local cross sections. In fact, given $a \in \tilde{\Sigma}_0(K)$, pick $\phi \in \pi^{-1}(a)$ and define $\sigma_a(o) := J_{(o,a)}(\phi)$ for $o \in V_a$. One can easily see that $\sigma_a : V_a \rightarrow B$ is a local cross section.

5 Principal bundles over posets

We now introduce the notion of a principal net bundle over a poset. This is a net bundle with a suitable action of a group. Many of the previous notions, like morphisms and connections, generalize straightforwardly by requiring equivariance. We study local trivializations, introduce transition functions and point out their main feature: they are locally constant.

Definition 5.1. A *principal net bundle* \mathcal{P} over a poset K is a net bundle (P, π, J, K) with a group G acting freely on the total space P on the right. The action R preserves the fibres, $\pi R_g = \pi$ for any $g \in G$, is transitive on the fibres, and is compatible with the net structure, namely $J R = R J$. We call G the *structure group* of \mathcal{P} . We denote the set of principal net bundles having structure group G by $\mathcal{P}(K, G)$.

In the sequel we adopt the following notation for the action of the structure group: $R_g(\psi) := \psi \cdot g$, with $\psi \in P$ and $g \in G$.

It is clear from this definition that the fibres of a principal net bundle are all isomorphic to the structure group. Furthermore, the relevant topology of the structure is the discrete one, since the order induced by the net structure on the total space is trivial when restricted to the fibres (see Section 4.1). Hence the topology induced on the fibres is discrete.

As an example, given a group G , consider the product net bundle $K \times G$ introduced in the previous section. Define

$$r_h(o, g) := (o, gh), \quad o \in K, \quad g, h \in G. \quad (4)$$

Clearly we have a principal net bundle that we call the *product principal net bundle*.

We now introduce principal morphisms and the category associated with principal net bundles.

Definition 5.2. Consider two principal net bundles $\mathcal{P}, \hat{\mathcal{P}} \in \mathcal{P}(K, G)$. A **morphism** f from $\hat{\mathcal{P}}$ into \mathcal{P} is an equivariant net bundle morphism $f : \hat{\mathcal{P}} \rightarrow \mathcal{P}$, namely $Rf = f\hat{R}$. We denote the set of morphisms from $\hat{\mathcal{P}}$ to \mathcal{P} by $(\hat{\mathcal{P}}, \mathcal{P})$.

The definition of morphisms involves only principal net bundles over the same poset K and having the same structure group. In Section 7.2, we will see how to connect principal net bundles having different structure groups. Now, given $\mathcal{P}, \hat{\mathcal{P}}, \tilde{\mathcal{P}} \in \mathcal{P}(K, G)$, let $f_1 \in (\mathcal{P}, \hat{\mathcal{P}})$ and $f_2 \in (\hat{\mathcal{P}}, \tilde{\mathcal{P}})$. Define

$$(f_2 f_1)(\phi) := f_2(f_1(\phi)), \quad \phi \in P. \quad (5)$$

It is easily seen that $f_2 f_1 \in (\mathcal{P}, \tilde{\mathcal{P}})$. The composition law (5.5) makes $\mathcal{P}(K, G)$ into a category, the category of *principal net bundles with structure group G* . We denote this by the same symbol as used to denote the set of objects. The identity $1_{\mathcal{P}}$ of $(\mathcal{P}, \mathcal{P})$ is the identity automorphism of \mathcal{P} . Since we are considering morphisms between principal bundles with the same structure group, it is easily seen, that any morphism $f \in (\hat{\mathcal{P}}, \mathcal{P})$ is indeed an *isomorphism*, namely there exists a morphism $f^{-1} \in (\mathcal{P}, \hat{\mathcal{P}})$ such that $f f^{-1} = 1_{\mathcal{P}}$ and $f^{-1} f = 1_{\hat{\mathcal{P}}}$. On these grounds, given $\hat{\mathcal{P}}, \mathcal{P} \in \mathcal{P}(K, G)$, and writing

$$\hat{\mathcal{P}} \cong \mathcal{P} \iff (\hat{\mathcal{P}}, \mathcal{P}) \neq \emptyset. \quad (6)$$

we endow $\mathcal{P}(K, G)$ of an equivalence relation \cong and we shall say that \mathcal{P}_1 and \mathcal{P} are *equivalent*. A principal net bundle $\mathcal{P} \in \mathcal{P}(K, G)$ is said to be *trivial* if it is equivalent to the principal product net bundle $K \times G$.

Following the scheme of the previous section, we now deal with the notion of a connection on a principal net bundle, and related notions like parallel transport, holonomy group, and flat connection. In particular we analyze this property from a global point of view, i.e. without using local trivializations.

Definition 5.3. A **connection** on a principal net bundle \mathcal{P} is a net bundle connection U of \mathcal{P} which is equivariant, namely

$$U(b)R = RU(b), \quad b \in \tilde{\Sigma}_1(K).$$

We denote the set of connections of a principal net bundle \mathcal{P} by $\mathcal{U}(K, \mathcal{P})$.

The notions of parallel transport along paths and flatness for generic net bundles admit a straightforward generalization to principal net bundles. The only important point to note is that if U is a connection of \mathcal{P} , and p is a path, then the parallel transport $U(p)$ is an equivariant map from $P_{\partial_1 p}$ to $P_{\partial_0 p}$. The *holonomy* and the *restricted holonomy* group of the connection U with respect to the base point $\psi \in P$ are defined by

$$\begin{aligned} H_U(\psi) &:= \{g \in G \mid \psi \cdot g = U(p)\psi, \quad \partial_0 p = \pi(\psi) = \partial_1 p\}; \\ H_U^0(\psi) &:= \{g \in G \mid \psi \cdot g = U(p)\psi, \quad \partial_0 p = \pi(\psi) = \partial_1 p, \quad p \sim \sigma_0 a\}. \end{aligned} \quad (7)$$

One sees that both $H_U(\psi)$ and $H_U^0(\psi)$ are indeed subgroups of G and that $H_U^0(\psi)$ is a normal subgroup of $H_U(\psi)$. This will be shown in Section 7.2 where we will deal with the cohomology of principal of net bundles and relate holonomy with reduction theory. We finally observe that a principal net bundle has a unique flat connection Z defined by equation (1).

Let U, \hat{U} be a pair of connections of the principal net bundles $\mathcal{P}, \hat{\mathcal{P}} \in \mathcal{P}(K, G)$ respectively. The set (\hat{U}, U) of the **morphisms** from \hat{U} to U is the subset of the morphisms $f \in (\hat{\mathcal{P}}, \mathcal{P})$ such that $f\hat{U} = Uf$. It is clearly an equivalence relation as any element of $(\hat{\mathcal{P}}, \mathcal{P})$ is invertible. It is worth observing that, given a principal net bundle \mathcal{P} , then $(Z, Z) = (\mathcal{P}, \mathcal{P})$ (Z is the flat connection on \mathcal{P}). The *category of connections of principal net bundles over K with structure group G* , is the category whose objects are connections on principal net bundles of $\mathcal{P}(K, G)$ and whose set of arrows are the corresponding morphisms. We denote this category by $\mathcal{U}(K, G)$. Furthermore, given $\mathcal{P} \in \mathcal{P}(K, G)$ we call the *category of connections on \mathcal{P}* , the full subcategory of $\mathcal{P}(K, G)$ whose set of objects is $\mathcal{U}(K, \mathcal{P})$. We denote this category by $\mathcal{U}(K, \mathcal{P})$ as for the corresponding set of objects.

Lemma 5.4. *The category $\mathcal{P}(K, G)$ is isomorphic to the full subcategory $\mathcal{U}_f(K, G)$ of $\mathcal{U}(K, G)$ whose objects are flat connections.*

Proof. Given $\mathcal{P} \in \mathcal{P}(K, G)$, define $F(\mathcal{P}) := Z$ where Z is the unique flat connection of \mathcal{P} . For any $f \in (\mathcal{P}, \mathcal{P}_1)$ define $F(f) = f$. Clearly $F : \mathcal{P}(K, G) \rightarrow \mathcal{U}_f(K, G)$ is a covariant functor. It is full because the set of morphisms $(\mathcal{P}, \mathcal{P}_1)$ equals the set of the morphisms of (Z, Z_1) . It is an isomorphism because for any Z , there obviously exists a principal net bundle \mathcal{P} , the bundle where Z is defined, such that $F(\mathcal{P}) = Z$. \square

We have seen that any net bundle is locally trivial (Proposition 4.5). We now show that principal net bundles are locally trivial too. This will allow us to investigate the local behaviour of trivialization maps, cross sections and morphisms. In particular we will point out the main feature of these local notions: all of them are, in a suitable sense, locally constant.

Proposition 5.5. *Any principal net bundle \mathcal{P} admits a local trivialization on the fundamental covering \mathcal{V}_0 .*

Proof. The proof is slightly different from the proof of Proposition 4.5, because equivariance is required. For any 0-simplex a , choose an element $\phi_a \in P_a$, and define

$$\theta_a(o, g) := J_{(o, a)}(\phi_a) \cdot g, \quad (o, g) \in V_a \times G. \quad (8)$$

As the action of the group is free $\theta_a : V_a \times G \rightarrow \pi^{-1}(V_a)$ is injective. Moreover, given $\phi \in \pi^{-1}(V_a)$ observe that $J_{(\pi(\phi), a)}(\phi_a)$ and ϕ belong to the same fibre. Since the action of G is transitive on the fibres, there exists $g_a(\phi) \in G$ such that $\phi = J_{(\pi(\phi), a)}(\phi_a) \cdot g_a(\phi)$. Then $\theta_a(\pi(\phi), g_a(\phi)) = J_{(\pi(\phi), a)}(\phi_a) \cdot g_a(\phi) = \phi$. This proves that θ_a is bijective. Now, given $b \in \Sigma_1(V_a)$, we have

$$J_b \theta_a(\partial_1 b, g) = J_b J_{\partial_1 b, a}(\phi_a) \cdot g = J_{\partial_0 b, a}(\phi_a) \cdot g = \theta_a(\partial_0 b, g) = \theta_a J_b(\partial_1 b, g).$$

Finally, it is clear that $R\theta_a = \theta_a r$. \square

As for net bundles, a *local trivialization* of a principal net bundle \mathcal{P} means a family of mappings $\theta := \{\theta_a\}$, $a \in K$, trivializing \mathcal{P} on the fundamental covering.

Consider a local trivialization θ of a principal net bundle \mathcal{P} . Given a pair of 0-simplices a, \tilde{a} , define

$$\theta_{\tilde{a}}^{-1} \theta_a(o, g) := (o, z_{\tilde{a}a}(o)g), \quad o \in V_{\tilde{a}a} \quad (9)$$

where $z_{\tilde{a}a}(o) \in G$. This definition is well posed because of the equivariance of θ_a . So we have a family $\{z_{\tilde{a}a}\}$ of functions $z_{\tilde{a}a} : V_{\tilde{a}a} \rightarrow G$ associated with the local trivialization θ .

Lemma 5.6. *Under the above assumptions and notation, $z_{a\tilde{a}} : V_a \rightarrow G$ are locally constant maps satisfying the cocycle identity*

$$z_{\hat{a}\tilde{a}}(o) z_{\tilde{a}a}(o) = z_{\hat{a}a}(o), \quad o \in V_{\hat{a}\tilde{a}a}.$$

Proof. Consider the 1-simplex (o_1, o) of the nerve. By the defining properties of a local trivialization (see proof of Proposition 5.5) we have $J_{(o_1, o)} \theta_a = J_{(o_1, o)} \theta_a$. This implies that $\theta_a J_{(o_1, o)} \theta_a^{-1} = \theta_{\tilde{a}} J_{(o_1, o)} \theta_{\tilde{a}}^{-1}$. Hence

$$(o_1, z_{\tilde{a}a}(o_1)) = \theta_{\tilde{a}}^{-1} \theta_a J_{(o_1, o)}(o, e) = J_{(o_1, o)} \theta_{\tilde{a}}^{-1} \theta_a(o, e) = (o_1, z_{\tilde{a}a}(o)),$$

and this proves that these maps are locally constant. The cocycle identity is obvious. \square

The functions $\{z_{\tilde{a}a}\}$ defined by equation (9) will be called *transition functions* of the principal net bundles. The cocycle identity says that transition functions can be interpreted in terms of the Čech cohomology of posets. This aspect will be developed in Section 8. Finally, the dependence of transition functions on the local trivialization will be discussed in Section 6.1 in terms of the associated 1-cocycles.

Consider a cross section $\sigma : V \rightarrow P$. We have already seen that σ is locally constant (see Subsection 4.3). Now, it is easily seen that, if $o \in V_{\tilde{a}a}$ and $\tilde{a}, a \in U$, then $s_{\tilde{a}}(o) = z_{\tilde{a}a}(o) s_a(a)$.

Lemma 5.7. *\mathcal{P} is trivial if, and only if, it admits a global section.*

Proof. (\Rightarrow) Given $f \in (K \times G, \mathcal{P})$, then $\sigma(o) := f(o, e)$ is a global cross section of \mathcal{P} . (\Leftarrow) Given a global section σ , define $f(o, g) := \sigma(o) \cdot g$ for any pair (o, g) . It is easily seen $f : K \times G \rightarrow P$ is an equivariant and fibre preserving bijective map. Furthermore, for any 1-simplex b of the nerve we have

$$\begin{aligned} f J_b(\partial_1 b, g) &= f(\partial_0 b, g) = \sigma(\partial_0 b) \cdot g = J_b(\sigma(\partial_1 b)) \cdot g \\ &= J_b(\sigma(\partial_1 b) \cdot g) = J_b f(\partial_1 b, g), \end{aligned}$$

completing the proof. \square

Morphisms between principal net bundles, like sections and local trivializations, are locally constant. Consider $f \in (\hat{\mathcal{P}}, \mathcal{P})$, and let $\hat{\theta}$ and θ be local trivializations of $\hat{\mathcal{P}}$ and \mathcal{P} respectively. Define

$$(o, f_a(o, g)) := \theta_a^{-1} f \hat{\theta}_a(o, g), \quad (o, g) \in V_a \times G. \quad (10)$$

This equation defines, for any 0-simplex a , a function $f_a : V_a \times G \rightarrow G$ enjoying the following properties:

Lemma 5.8. *Under the above notation and assumptions, the function $f_a : V_a \times G \rightarrow G$ is locally constant and $f_a(o, g) = f_a(o, e)g$, for any $(o, g) \in V_a \times G$.*

Proof. Given $b \in \Sigma_1(V_a)$. Since $f J f = f \hat{J}$, we have

$$\begin{aligned} (\partial_0 b, f_a(\partial_0 b, e)) &= \theta_a^{-1} f \hat{\theta}_a(\partial_0 b, e) = \theta_a^{-1} f \hat{\theta}_a j_b(\partial_1 b, e) \\ &= \theta_a^{-1} f \hat{J}_b \hat{\theta}_a(\partial_1 b, e) = \theta_a^{-1} J_b f \hat{\theta}_a(\partial_1 b, e) \\ &= j_b \theta_a^{-1} f \hat{\theta}_a(\partial_1 b, e) = j_b(\partial_1 b, f_a(\partial_1 b, e)) \\ &= (\partial_0 b, f_a(\partial_1 b, e)). \end{aligned}$$

Hence f_a is locally constant. Equivariance of f completes the proof. \square

6 Cohomological representation and equivalence

Because of the lack of a differential structure, we replace the differential calculus of forms with a cohomology taking values in the structure groups of principal net bundles. In the present section we show that the theory of principal bundles and connections over posets described in the previous section, admits an equivalent description in terms of a non-Abelian cohomology of posets [10]. We will construct mappings associating to geometrical objects like principal net bundles and connections, the corresponding cohomological objects: 1-cocycles and connections 1-cochains. We refer the reader to the Appendix for notation and a brief description of the results of the cited paper.

6.1 Cohomological representation

Consider a connection U on a principal net bundle \mathcal{P} . Let θ be a local trivialization of \mathcal{P} . Given a 1-simplex b of $\hat{\Sigma}_1(K)$, define

$$(\partial_0 b, \Gamma_\theta(U)(b) \cdot g) := \theta_{\partial_0 b}^{-1} U(b) \theta_{\partial_1 b}(\partial_1 b, g), \quad (11)$$

for any $g \in G$. By equivariance, this definition is well posed. This equation associates a 1-cochain $\Gamma_\theta(U) : \tilde{\Sigma}_1(K) \rightarrow G$ to the connection U .

Proposition 6.1. *Given $\mathcal{P} \in \mathcal{P}(K, G)$, let θ be a local trivialization of \mathcal{P} . Then the following assertions hold:*

- (i) $\Gamma_\theta(U) \in U^1(K, G)$ for any $U \in \mathcal{U}(K, \mathcal{P})$;
- (ii) $\Gamma_\theta(Z) \in Z^1(K, G)$;
- (iii) $\Gamma_\theta(U) \in U^1(K, \Gamma_\theta(Z))$ for any $U \in \mathcal{U}(K, \mathcal{P})$.

Proof. It is convenient to introduce a new notation. Given two 0-simplices a and a_1 and an element $g \in G$, let $\ell_{a,a_1}(g) : a_1 \times G \rightarrow a \times G$ be defined by $\ell_{a,a_1}(g)(a_1, h) := (a, gh)$ for any $h \in G$. Then observe that equation (11) can be rewritten as

$$\ell_{\partial_0 b, \partial_1 b}(\Gamma_\theta(U)(b)) = \theta_{\partial_0 b}^{-1} U(b) \theta_{\partial_1 b} . \quad (12)$$

Now, given $U \in \mathcal{U}(K, \mathcal{P})$ and a 1-simplex b , observe that

$$\begin{aligned} \ell_{\partial_1 b, \partial_0 b}(\Gamma_\theta(U)(\bar{b})) &= \theta_{\partial_1 b}^{-1} U(b)^{-1} \theta_{\partial_0 b} \\ &= (\theta_{\partial_0 b}^{-1} U(b) \theta_{\partial_1 b})^{-1} = \ell_{\partial_1 b, \partial_0 b}(\Gamma_\theta(U)(b)^{-1}) . \end{aligned}$$

Hence $\Gamma_\theta(U)(\bar{b}) = \Gamma_\theta(U)(b)^{-1}$. If b is a 1-simplex of the nerve, then

$$\ell_{\partial_0 b, \partial_1 b}(\Gamma_\theta(U)(b)) = \theta_{\partial_0 b}^{-1} J_{\partial_0 b, \partial_1 b} \theta_{\partial_1 b} = \theta_{\partial_0 b}^{-1} Z(b) \theta_{\partial_1 b} = \ell_{\partial_0 b, \partial_1 b}(\Gamma_\theta(Z)(b)) .$$

Hence $\Gamma_\theta(U)$ is equal to $\Gamma_\theta(Z)$ on 1-simplices of the nerve. It remains to show that $\Gamma_\theta(Z)$ is a 1-cocycle. Given a 2-simplex c we have

$$\begin{aligned} \ell_{\partial_{00} c, \partial_{12} c}(\Gamma_\theta(Z)(\partial_0 c) \Gamma_\theta(Z)(\partial_2 c)) &= \\ &= \theta_{\partial_{00} c}^{-1} Z(\partial_0 c) \theta_{\partial_{10} c} \theta_{\partial_{02} c}^{-1} Z(\partial_2 c) \theta_{\partial_{12} c} \\ &= \theta_{\partial_{01} c}^{-1} Z(\partial_1 c) \theta_{\partial_{11} c} = \theta_{\partial_{00} c}^{-1} Z(\partial_1 c) \theta_{\partial_{12} c} \\ &= \ell_{\partial_{00} c, \partial_{12} c}(\Gamma_\theta(Z)(\partial_1 c)) , \end{aligned}$$

and this completes the proof. \square

Sometimes, when no confusion is possible, we will adopt the following notation

$$\begin{aligned} u_\theta &:= \Gamma_\theta(U), \quad U \in \mathcal{U}(\mathcal{P}), \\ z_\theta &:= \Gamma_\theta(Z), \quad Z \in \mathcal{U}(\mathcal{P}), \end{aligned} \quad (13)$$

where Z is the flat connection of \mathcal{P} . Since Z is unique, we will refer to z_θ as the *bundle cocycle* of \mathcal{P} . Furthermore, we will call u_θ the *connection cochain* of the connection U . We shall see in Proposition 6.3 how these objects depend on the choice of local trivialization. Notice that the assertion (iii) of the above proposition, says that for any connection $U \in \mathcal{U}(K, \mathcal{P})$, z_θ is the 1-cocycle induced by u_θ (see Appendix).

The next result shows the relation between the bundle cocycle and transition functions of the bundle.

Lemma 6.2. *Given a principal net bundle \mathcal{P} , let θ be a local trivialization and $\{z_{a\bar{a}}\}$ the corresponding family of transition functions. Then*

$$z_\theta(b) = z_{\partial_0 b, \partial_1 b}(|b|) , \quad b \in \tilde{\Sigma}_1(K).$$

Proof. Using the definition of transition functions, we have

$$\begin{aligned} \ell_{|b|, |b|}(z_{\partial_0 b, \partial_1 b}(|b|)) &= \theta_{\partial_0 b}^{-1} \theta_{\partial_1 b} \\ &= \theta_{\partial_0 b}^{-1} \theta_{\partial_1 b} J_{(|b|, \partial_1 b)} J_{(|b|, \partial_1 b)}^{-1} \\ &= \theta_{\partial_0 b}^{-1} J_{(|b|, \partial_1 b)} \theta_{\partial_1 b} J_{(|b|, \partial_1 b)}^{-1} \\ &= \theta_{\partial_0 b}^{-1} J_{(|b|, \partial_0 b)} J_{(|b|, \partial_0 b)}^{-1} J_{(|b|, \partial_1 b)} \theta_{\partial_1 b} J_{(|b|, \partial_1 b)}^{-1} \\ &= J_{(|b|, \partial_0 b)} \theta_{\partial_0 b}^{-1} Z(b) \theta_{\partial_1 b} J_{(|b|, \partial_1 b)}^{-1} \\ &= J_{(|b|, \partial_0 b)} \ell_{\partial_0 b, \partial_1 b}(z_\theta(b)) J_{(|b|, \partial_1 b)}^{-1} \\ &= \ell_{|b|, |b|}(z_\theta(b)) , \end{aligned}$$

and this completes the proof. \square

The mapping Γ_θ defines a correspondence between connections and connection 1-cochains. Moreover, as a principal net bundle \mathcal{P} has a unique flat connection Z , the relation $\Gamma_\theta(Z) = z_\theta$ gives a correspondence between principal net bundles and 1-cocycles. We will show that these correspondences are equivalences of categories. To this end, we extend this map to morphisms.

Consider $\hat{\mathcal{P}}, \mathcal{P} \in \mathcal{P}(K, G)$, and let $\hat{\theta}$ and θ be, respectively, local trivializations of $\hat{\mathcal{P}}$ and \mathcal{P} . Given $f \in (\hat{\mathcal{P}}, \mathcal{P})$, define

$$(a, \Gamma_{\theta\hat{\theta}}(f)_a g) := (\theta_a^{-1} f \hat{\theta}_a)(a, g) , \quad (14)$$

for any 0-simplex a and for any $g \in G$. Clearly, $\Gamma_{\theta\hat{\theta}}(f) : \tilde{\Sigma}_0(K) \rightarrow G$.

Proposition 6.3. *Let $\hat{\mathcal{P}}, \mathcal{P} \in \mathcal{P}(K, G)$, and let $\hat{\theta}$ and θ be, respectively, local trivializations. Then, given $\hat{U} \in \mathcal{U}(K, \hat{\mathcal{P}})$ and $U \in \mathcal{U}(K, \mathcal{P})$, we have*

$$\Gamma_{\theta\hat{\theta}}(f) \in (\hat{u}_{\hat{\theta}}, u_{\theta}), \quad f \in (\hat{U}, U).$$

In particular if $f \in (\hat{\mathcal{P}}, \mathcal{P})$, then $\Gamma_{\theta\hat{\theta}}(f) \in (\hat{z}_{\hat{\theta}}, z_{\theta})$.

Proof. First of all observe that the equation (14) is equivalent to

$$\ell_{aa}(\Gamma_{\theta\hat{\theta}}(f)_a) = \theta_a^{-1} f \hat{\theta}_a. \quad (15)$$

Now, given a 1-simplex b , we have

$$\begin{aligned} \ell_{\partial_0 b, \partial_1 b}(\Gamma_{\theta\hat{\theta}}(f)_{\partial_0 b} \hat{u}_{\hat{\theta}}(b)) &= \theta_{\partial_0 b}^{-1} f \hat{\theta}_{\partial_0 b}^{-1} \hat{U}(b) \hat{\theta}_{\partial_1 b} \\ &= \theta_{\partial_0 b}^{-1} U(b) \theta_{\partial_1 b} \theta_{\partial_1 b}^{-1} f \hat{\theta}_{\partial_1 b} = \ell_{\partial_0 b, \partial_1 b}(u_{\theta}(b) \Gamma_{\theta\hat{\theta}}(f)_{\partial_1 b}), \end{aligned}$$

and this completes the proof. \square

This result provides the formula for changing a local trivialization. In fact let θ and ϕ be two local trivialization of the bundle \mathcal{P} . Then

$$\Gamma_{\theta\phi}(1_{\mathcal{P}}) \in (u_{\phi}, u_{\theta}), \quad U \in \mathcal{U}(K, \mathcal{P}). \quad (16)$$

Hence, changing the local trivializations leads to equivalent connection 1-cochains.

We now show that Γ_{θ} is invertible. To this end, given $u \in U^1(K, z_{\theta})$, $\hat{u} \in U^1(K, \hat{z}_{\hat{\theta}})$, and $f \in (u, \hat{u})$, define

$$\begin{aligned} \Upsilon_{\theta}(u)(\psi) &:= (\theta_{\partial_0 b} \ell_{\partial_0 b, \partial_1 b}(u(b)) \theta_{\partial_1 b}^{-1})(\psi), \quad \psi \in \pi_1^{-1}(\partial_1 b), \\ \Upsilon_{\theta\hat{\theta}}(f)(\psi) &:= (\theta_a \ell_{a, a}(f_a) \hat{\theta}_a^{-1})(\psi), \quad \psi \in \pi_1^{-1}(a). \end{aligned} \quad (17)$$

for any $b \in \tilde{\Sigma}_1(K)$ and $a \in \tilde{\Sigma}_0(K)$.

Proposition 6.4. *Let $\mathcal{P}, \hat{\mathcal{P}}$ be principal net bundles with local trivializations $\theta, \hat{\theta}$, respectively. Then*

- (i) $\Upsilon_{\theta} : U^1(K, z_{\theta}) \rightarrow \mathcal{U}(K, \mathcal{P})$ is the inverse of Γ_{θ} .
- (ii) Given $u \in U^1(K, z_{\theta})$ and $\hat{u} \in U^1(K, \hat{z}_{\hat{\theta}})$, then $\Upsilon_{\theta\hat{\theta}} : (\hat{u}, u) \rightarrow (\Upsilon_{\hat{\theta}}(\hat{u}), \Upsilon_{\theta}(u))$ is the inverse of $\Gamma_{\theta\hat{\theta}}$

Proof. (i) Clearly $\Upsilon_\theta(u)(b)$ is a bijection from $\pi^{-1}(\partial_1 b)$ into $\pi^{-1}(\partial_0 b)$. Given the reverse \bar{b} of b we have

$$\Upsilon_\theta(u)(\bar{b}) = \theta_{\partial_0 \bar{b}} \ell_{\partial_0 \bar{b}, \partial_1 \bar{b}}(u(\bar{b})) \theta_{\partial_1 \bar{b}}^{-1} = \theta_{\partial_1 b} \ell_{\partial_1 b, \partial_0 b}(u(b)^{-1}) \theta_{\partial_0 b}^{-1} = \Upsilon_\theta(u)(b)^{-1}.$$

Let b be a 1-simplex of the nerve. Since $u(b) = z_\theta(b)$, by (12), we have

$$\Upsilon_\theta(u)(b) = \theta_{\partial_0 b} \ell_{\partial_0 b, \partial_1 b}(z_\theta(b)) \theta_{\partial_1 b}^{-1} = \theta_{\partial_0 b} \theta_{\partial_0 b}^{-1} Z(b) \theta_{\partial_1 b} \theta_{\partial_1 b}^{-1} = J_{\partial_0 b, \partial_1 b}.$$

Hence $\Upsilon_\theta(u)$ is a connection of \mathcal{P} . The identities $\Gamma_\theta(\Upsilon_\theta(u)) = u$ and $\Upsilon_\theta(\Gamma_\theta(U)) = U$ follow straightforwardly from the definitions of Γ_θ and Υ_θ . (ii) Given a 1-simplex b , we have

$$\begin{aligned} \Upsilon_{\theta\hat{\theta}}(f) \Upsilon_{\hat{\theta}}(u)(b) &= \theta_{\partial_0 b} \ell_{\partial_0 b, \partial_0 b}(f_{\partial_0 b}) \ell_{\partial_0 b, \partial_1 b}(u(b)) \theta_{\partial_1 b}^{-1} \\ &= \theta_{\partial_0 b} \ell_{\partial_0 b, \partial_1 b}(f_{\partial_0 b} u(b)) \hat{\theta}_{\partial_1 b}^{-1} \\ &= \theta_{\partial_0 b} \ell_{\partial_0 b, \partial_1 b}(u(b) f_{\partial_1 b}) \hat{\theta}_{\partial_1 b}^{-1} \\ &= \Upsilon_\theta(u)(b) \Upsilon_{\theta\hat{\theta}}(f) \end{aligned}$$

This, in particular, implies that $\Upsilon_{\theta\hat{\theta}}(f) \hat{J}_b = J_b f$ for any $b \in \Sigma_1(K)$, as any connection coincides with the flat connection when restricted to the nerve. Equivariance is obvious. \square

The previous results point to an equivalence between the category of connections of a principal net bundle and the category of connection 1-cochains of the corresponding bundle cocycle.

Theorem 6.5. *Given a principal net bundle \mathcal{P} and a local trivialization θ . Then the categories $\mathcal{U}(K, \mathcal{P})$ and $U^1(K, z_\theta)$ are isomorphic.*

Proof. Using Propositions 6.1 and 6.2, it is easily seen that the mappings $\mathcal{U}(K, \mathcal{P}) \ni U \rightarrow u_\theta \in U^1(K, z_\theta)$ and $(U, \tilde{U}) \ni f \rightarrow \Gamma_{\theta\hat{\theta}}(f) \in (u_\theta, \tilde{u}_\theta)$, define a covariant functor from $U^1(K, z_\theta)$ to $\mathcal{U}(K, \mathcal{P})$. Conversely, by Proposition 6.4, the mappings $U^1(K, z_\theta) \ni u \rightarrow \Upsilon_\theta(u) \in \mathcal{U}(K, \mathcal{P})$ and $(u, \tilde{u}) \ni f \rightarrow \Upsilon_{\theta\hat{\theta}}(f) \in (\Upsilon_\theta(u), \Upsilon_\theta(\tilde{u}))$ define a covariant functor from $U^1(K, z_\theta)$ to $\mathcal{U}(K, \mathcal{P})$. Proposition 6.4 implies that these two functors are isomorphisms of categories. \square

6.2 General equivalence

We want to prove that any 1-cocycle z gives rise to a principal net bundle and any connection 1-cochain $u \in U^1(K, z)$ to a connection of the principal

net bundle associated with z . We will prove this by a series of results. The first one imitates the reconstruction of a principal bundle from its transition functions.

Proposition 6.6. *Given any 1-cocycle $z \in Z^1(K, G)$ there is a principal net bundle \mathcal{P}_z , with structure group G , and a local trivialization θ_z such that $\Gamma_{\theta_z}(Z) = z$, where Z is the flat connection of \mathcal{P}_z .*

Proof. Given $z \in Z^1(K, G)$, for any pair a, \tilde{a} , define

$$z_{a\tilde{a}}(o) := z(o; a, \tilde{a}), \quad o \in V_{a\tilde{a}}$$

where $(o; a, \tilde{a})$ is the 1-simplex with support o , 1-face \tilde{a} and 0-face a . It is easily seen that the functions $z_{a\tilde{a}} : V_{a\tilde{a}} \rightarrow G$ satisfy all the properties of transition functions (Lemma 5.6). Let

$$P := \bigcup_{a \in \tilde{\Sigma}_0(K)} V_a \times G \times \{a\}$$

Any element of P is of the form (o, g, a) where $a \in \tilde{\Sigma}_0(K)$, $o \in V_a$ and $g \in G$. Equip P with the equivalence relation

$$(o, g, a) \sim_z (o_1, g_1, a_1) \iff o = o_1 \in V_{aa_1} \text{ and } g = z_{aa_1}(o)g_1.$$

Denote the equivalence class associated with the element (o, g, a) by $[o, g, a]_z$, and define

$$P_z := \{[o, g, a]_z \mid a \in \tilde{\Sigma}_0(K), o \in V_a, g \in G\}$$

There is a surjective map $\pi_z : P_z \rightarrow K$ defined by $\pi_z([o, g, a]_z) := o$ with $[o, g, a]_z \in P_z$. Given $b \in \Sigma_1(K)$, define

$$J_{zb}[\partial_1 b, g, a]_z := [\partial_0 b, g, a]_z.$$

This definition is well posed. In fact, if $(\partial_1 b, g, a) \sim_z (\partial_1 b, g_1, a_1)$, then $g = z_{aa_1}(\partial_1 b) \cdot g_1$. Since $\partial_1 b \leq \partial_0 b$ we have $z_{aa_1}(\partial_1 b) = z_{aa_1}(\partial_0 b)$. Hence $(\partial_0 b, g, a) \sim_z (\partial_0 b, g_1, a_1)$. J_{zb} is a bijection and the fibres are isomorphic to G . Define

$$R_{zh}[o, g, a]_z := [o, gh, a]_z, \quad h \in G.$$

This is a free right action on P_z because $[o, g, a]_z = [o, gh, a]_z$ if, and only if, $g = gh$ and is transitive on the fibres. Given $b \in \Sigma_1(K)$ we have

$$J_{zb} R_{zh}[\partial_1 b, g, a]_z = [\partial_0 b, gh, a]_z = R_{zh}[\partial_0 b, g, a]_z = R_{zh} J_{zb}[\partial_1 b, g, a]_z.$$

So the data $\mathcal{P}_z := (P_z, \pi_z, J_z, G, K)$ is a principal net bundle over K with structure group G . Given a 0-simplex a , let

$$\theta_{za}(o, g) := J_{z(o,a)}([a, e, a]_z) \cdot g, \quad (o, g) \in V_a \times G.$$

Observe that

$$\theta_{za_1}^{-1} \theta_{za}(o, g) = \theta_{za_1}^{-1} [o, g, a]_z = \theta_{za_1}^{-1} [o, z_{aa_1}(o)g, a_1]_z = (o, z_{aa_1}(o)g).$$

The proof then follows by Lemma 6.2. \square

Thus the principal net bundle \mathcal{P}_z has bundle cocycle z . Now, since any connection 1-cochain induces a 1-cocycle, we expect that any connection 1-cochain comes from a connection on a principal net bundle. This is the content of the next result.

For any $\mathcal{P} \in \mathcal{P}(K, G)$, choose a local trivialization θ of \mathcal{P} and denote such a *choice* by the symbol $\underline{\theta}$.

Theorem 6.7. *The following assertions hold:*

- (i) *The categories $Z^1(K, G)$ and $\mathcal{P}(K, G)$ are equivalent;*
- (ii) *The categories $U^1(K, G)$ and $\mathcal{U}(K, G)$ are equivalent.*

Proof. We first prove the assertion (ii). Define

$$\begin{aligned} \Gamma_{\underline{\theta}}(U) &:= \Gamma_{\theta}(U), \quad U \in \mathcal{U}(K, \mathcal{P}); \\ \Gamma_{\underline{\theta}}(f) &:= \Gamma_{\theta\hat{\theta}}(f), \quad f \in (\hat{U}, U), \end{aligned} \tag{18}$$

where $U \in \mathcal{U}(K, \mathcal{P})$, and $U_1 \in \mathcal{U}(K, \hat{\mathcal{P}})$. It easily follows from Propositions 6.1 and 6.3 that $\Gamma_{\underline{\theta}} : \mathcal{U}(K, G) \rightarrow U^1(K, G)$ is a covariant functor. Conversely, observe that $U^1(K, G)$ is a disjoint union of the $U^1(K, z)$ as z varies in $Z^1(K, G)$ (see Appendix), and define

$$\begin{aligned} \Upsilon(u) &:= \Upsilon_{\theta_z}(u), \quad u \in U^1(K, z); \\ \Upsilon(f) &:= \Upsilon_{\theta_z\theta_z}(f), \quad f \in (u, \hat{u}), \end{aligned} \tag{19}$$

where $u \in U^1(K, z)$ and $\hat{u} \in U^1(\hat{z})$, and θ_z is the local trivialization of \mathcal{P}_z in Proposition 6.6. By Proposition 6.4, Υ is a covariant functor.

We now prove that the pair $\Gamma_{\underline{\theta}}, \Upsilon$ is an equivalence of categories. To this end, consider a connection U on \mathcal{P} and the connection $\Upsilon\Gamma_{\underline{\theta}}(U) = \Upsilon_{\theta_{z_\theta}}(u_\theta)$ defined on the principal net bundle \mathcal{P}_{z_θ} reconstructed from the bundle cocycle z_θ . Define

$$x(U)[o, g, a]_{z_\theta} := \theta_a(o, g), \quad (o, g) \in V_a \times G.$$

This definition is well posed. In fact if $(o, g, a) \sim_{z_\theta} (o, g_1, a_1)$, that is, if $g = z_{\theta_{aa_1}}(o)g_1$, then $\theta_a(o, g) = \theta_a(o, z_{\theta_{aa_1}}(o)g_1) = \theta_a \theta_a^{-1} \theta_{a_1}(o, g_1) = \theta_{a_1}(o, g_1)$, where $\{z_{\theta_{aa_1}}\}$ are the transition functions associated with z_θ . The definitions of $\Upsilon_{\theta_{z_\theta}}$ and $\Gamma_\theta(U)$ imply that x is a natural isomorphism between $\Upsilon \Gamma_\theta$ and $1_{\mathcal{U}(K, G)}$. Conversely given a $z \in Z^1(K, G)$, define

$$y(u) := \Gamma_{\theta_z \theta'}(\mathbf{I}_{\mathcal{P}_z}), \quad u \in U^1(K, z),$$

where θ' denotes the local trivialization of \mathcal{P}_z associated with $\underline{\theta}$, and θ_z that defined in Proposition 6.6. Proposition 6.3 implies that y is a natural isomorphism between $\Gamma_\theta \Upsilon$ and $1_{U^1(K, G)}$. (i) follows from (ii) by observing that $\Gamma_\theta : \mathcal{U}_f(K, G) \rightarrow Z^1(K, G)$ and $\Upsilon : Z^1(K, G) \rightarrow \mathcal{U}_f(K, G)$ and that $\mathcal{U}_f(K, G)$ is isomorphic to $\mathcal{P}(K, G)$ (Lemma 5.4). \square

7 Curvature and reduction

The equivalence between the geometrical and the cohomological description of principal bundles over posets allows us to analyze connections further. In particular, we will discuss: the curvature of a connection, relating it to flatness; the existence of non-flat connections; the reduction of principal net bundles and connections.

7.1 Curvature

We now introduce the curvature of a connection and use the notation of (13) to indicate the connection cochain and the bundle cocycle.

Definition 7.1. *Let $\mathcal{P} \in \mathcal{P}(K, G)$ have a local trivialization θ . The **curvature cochain** of a connection U on \mathcal{P} is the 2-coboundary $du_\theta \in B^2(K, G)$ of the connection cochain $u_\theta \in U^1(K, G)$, namely*

$$\begin{aligned} (du_\theta)_1(b) &:= (\iota, \text{ad}(u_\theta(b)), & b \in \tilde{\Sigma}_1(K), \\ (du_\theta)_2(c) &:= (w_{u_\theta}(c), \text{ad}(u_\theta(\partial_1 c)), & c \in \tilde{\Sigma}_2(K), \end{aligned}$$

where $w_{u_\theta} : \tilde{\Sigma}_2(K) \rightarrow G$ is defined by

$$w_{u_\theta}(c) := u_\theta(\partial_0 c) u_\theta(\partial_2 c) u_\theta(\partial_1 c)^{-1}, \quad c \in \tilde{\Sigma}_2(K),$$

and $\text{ad}(g)$ denotes the inner automorphism of G defined by $g \in G$.

Some observations are in order. *First*, this definition does not depend on the choice of local trivialization. To be precise, if θ_1 is another local trivialization of \mathcal{P} , the two coboundaries du_θ and du_{θ_1} are equivalent objects in the category of 2-cochains (see [10, 8]). However, we prefer not to verify this as it would involve the heavy machinery of non-Abelian cohomology. *Secondly*, in this framework, the Bianchi identity for a connection 1-cochain u corresponds to the 2-cocycle identity, namely

$$w_u(\partial_0 d) w_u(\partial_2 d) = \text{ad}(u(\partial_{01} d))(w_u(\partial_3 d)) w_u(\partial_1 d), \quad (20)$$

for any 3-simplex d . *Thirdly*, in [10] a connection u is defined to be flat whenever its curvature is trivial. This amounts to saying that $w_u(c) = e$ for any 2-simplex c . Now as the mapping Γ_θ is invertible, a connection U is flat if, and only if, its curvature is trivial.

We now draw on two consequence of this definition of curvature and of the equivalence between the geometrical and the cohomological description of principal bundles over posets.

Corollary 7.2. *There is, up to equivalence, a 1-1 correspondence between flat connections of principal net bundles over K with structure group G and group homomorphisms of the fundamental group $\pi_1(K)$ of the poset with values in the group G .*

Proof. By Theorem 6.7 the proof follows from [10, Corollary 4.6]. \square

This is the poset version of a classical result of the theory of fibre bundles over manifolds (see [5]). Note that this corollary and Theorem 6.7(i) imply that principal net bundles over a simply connected poset are trivial. The existence of nonflat connections is the content of the next result.

Corollary 7.3. *Assume that K is a pathwise connected but not totally directed poset, and let G be a nontrivial group. Then, the set of nonflat connections of a principal net bundle $\mathcal{P} \in \mathcal{P}(K, G)$ is never empty.*

Proof. By Theorem 6.7, the proof follows from [10, Theorem 4.25]. \square

7.2 Reduction theory

The cohomological representation allows us to compare principal bundles having different structure groups. Any group homomorphism $\gamma : H \rightarrow G$ defines a covariant functor $\gamma \circ : C^1(K, H) \rightarrow C^1(K, G)$ (see Appendix). This functor maps 1-cocycles and connection 1-cochains with values in H into

1-cocycles and connection 1-cochains with values in G . Moreover $\gamma \circ$ turns out to be an isomorphism whenever γ is a group isomorphism. So using the functors Υ and $\Gamma_{\underline{\theta}}$ introduced in the proof of Theorem 6.7, we have that

$$\Upsilon \gamma \circ \Gamma_{\underline{\theta}} : \mathcal{U}(K, H) \rightarrow \mathcal{U}(K, G)$$

and, clearly,

$$\Upsilon \gamma \circ \Gamma_{\underline{\theta}} : \mathcal{P}(K, H) \rightarrow \mathcal{P}(K, G)$$

are covariant functors and even equivalences of categories when $\gamma \circ$ is a group isomorphism.

We now define reducibility both for connections and principal net bundles.

Definition 7.4. A *connection* U on a principal net bundle $\mathcal{P} \in \mathcal{P}(K, G)$ is said to be **reducible**, if there is a proper subgroup $H \subset G$, a principal net bundle $\tilde{\mathcal{P}} \in \mathcal{P}(K, H)$ and a connection $\tilde{U} \in \mathcal{U}(K, \tilde{\mathcal{P}})$ such that

$$\Upsilon_{\theta_{\tilde{z}_\phi}} \iota_{G,H} \Gamma_\phi(\tilde{U}) \cong U . \quad (21)$$

for a local trivialization ϕ of $\tilde{\mathcal{P}}$. A **principal net bundle** \mathcal{P} is **reducible** if its flat connection is reducible.

In this definition $\iota_{G,H} : H \rightarrow G$ is the inclusion mapping. So, recalling the notation (13), $\Upsilon_{\theta_{\tilde{z}_\phi}} \iota_{G,H} \Gamma_\phi(\tilde{U})$ is the reconstruction of a connection from the connection 1-cochain $\tilde{u}_\phi \in U^1(K, H)$ considered as taking values in the larger group G . $\theta_{\tilde{z}_\phi}$ is the local trivialization of the principal net bundle reconstructed from the bundle cocycle \tilde{z}_ϕ (see Proposition 6.6) considered as a cocycle with values in G . Moreover, the bundle cocycle \tilde{z}_ϕ is equal to the 1-cocycle induced by \tilde{u}_ϕ (Proposition 6.1). Finally, by Theorem 6.7 this definition is independent of the local trivialization.

We now derive an analogue of the Ambrose-Singer theorem for principal bundles and connections over posets. This has already been done in the cohomological approach [10, Theorem 4.28]. The next two lemmas are needed for extending this result to fibre bundles.

Lemma 7.5. A connection U on $\mathcal{P} \in \mathcal{P}(K, G)$ is reducible if, and only if, the connection cochain u_θ of a local trivialization θ is reducible.

Proof. (\Rightarrow) Assume that u_θ is reducible. Then there is a proper subgroup H of G and a connection 1-cochain $v \in U^1(K, H)$ such that $v \cong u_\theta$ in the category $U^1(K, G)$. By virtue of Theorem 6.7, there is a connection

$\tilde{U} \in \mathcal{U}(K, \tilde{\mathcal{P}})$ with $\tilde{\mathcal{P}} \in \mathcal{P}(K, H)$ such that $\tilde{u}_{\tilde{\theta}} = v$ for a local trivialization $\tilde{\theta}$ of $\tilde{\mathcal{P}}$. Hence $\tilde{u}_{\tilde{\theta}} \cong u_{\theta}$ in $U^1(K, G)$. The corresponding bundle cocycles $\tilde{z}_{\tilde{\theta}}$ and z_{θ} are then equivalent in $Z^1(K, G)$ [10, Lemma 4.14]. By Proposition 6.4(ii) we have

$$\Upsilon_{\theta_{\tilde{z}_{\tilde{\theta}}}}(\tilde{u}_{\tilde{\theta}}) \cong \Upsilon_{\theta_{z_{\theta}}}(u_{\theta}) = U.$$

This completes the proof. The implication (\Leftarrow) follows similarly. \square

We next relate the holonomy groups of a connection to those of the corresponding connection cochain. The holonomy group $H_u(a)$ and the restricted holonomy group $H_u^0(a)$, based on a 0-simplex a , of a connection 1-cochain u of $U^1(K, G)$ are defined, respectively, by $H_u(a) = \{u(p) \in G \mid \partial_0 p = a = \partial_1 p\}$ and $H_u^0(a) = \{u(p) \in G \mid \partial_0 p = a = \partial_1 p, p \sim \sigma_0 a\}$ (see [10]). Then we have the following

Lemma 7.6. *Let $U \in \mathcal{U}(\mathcal{P})$ and $\psi \in \mathcal{P}$ with $\pi(\psi) = a$. Given a local trivialization θ of \mathcal{P} , $H_U(\psi)$ and $H_{u_{\theta}}(a)$ are conjugate subgroups of G . The same assertion holds for the restricted holonomy groups.*

Proof. If $g \in H_U(\psi)$, then there is a loop p such that $\partial_0 p = a = \partial_1 p$ and $R_g(\psi) = U(p)\psi$. Now, let $(a, g_{\psi}) := \theta_a^{-1}\psi$, then

$$\begin{aligned} (a, g_{\psi}g) &= r(g)\theta_a^{-1}\psi = \theta_a^{-1}R_g\psi = \theta_a^{-1}U(p)\psi \\ &= \theta_a^{-1}U(b_n) \cdots U(b_1)\psi \\ &= \theta_a^{-1}U(b_n)\theta_{\partial_1 b_n}^{-1}\theta_{\partial_1 b_n}^{-1} \cdots U(b_2)\theta_{\partial_0 b_1}^{-1}\theta_{\partial_0 b_1}^{-1}U(b_1)\theta_a\theta_a^{-1}\psi \\ &= \ell_{a, \partial_1 b_n}(u_{\theta}(b_n)) \cdots \ell_{\partial_0 b_1, a}(u_{\theta}(b_1))\theta_a^{-1}\psi \\ &= \ell_{a, a}(u_{\theta}(p))\theta_a^{-1}\psi \\ &= (a, u_{\theta}(p)g_{\psi}) \end{aligned}$$

The mapping $H_U(\psi) \ni g \rightarrow g_{\psi}gg_{\psi}^{-1} \in H_{u_{\theta}}(a)$ is clearly a group isomorphism. \square

This result and [10, Lemma 4.27] lead to the following conclusions: the holonomy groups are subgroups of G ; the restricted group is a normal subgroup of the unrestricted group; a change of the base ψ leads to conjugate holonomy groups and equivalent connections lead to isomorphic holonomy groups.

Finally, we have the analogue of the Ambrose-Singer theorem for principal net bundles.

Theorem 7.7. *Given $\mathcal{P} \in \mathcal{P}(K, G)$, let $\psi \in \mathcal{P}$. Let $U \in \mathcal{U}(K, \mathcal{P})$. Then*

(i) \mathcal{P} is reducible to a principal net bundle $\tilde{\mathcal{P}} \in \mathcal{P}(K, H_U(\psi))$.

(ii) U is reducible to a connection $\tilde{U} \in \mathcal{U}(K, \tilde{\mathcal{P}})$.

Proof. The proof follows from the previous lemmas and [10, Theorem 4.28]. \square

8 Čech cohomology and flat bundles

In the previous sections, we introduced the transition functions of a (principal) net bundle, and verified that they satisfy cocycle identities analogous to those encountered in the Čech cohomology of a topological space. This allows us to define the *Čech cohomology* of a poset, showing it to be equivalent to the net cohomology of the poset. Then, we specialize our discussion to the poset of open, contractible subsets of a manifold. In this case, the above constructions yield the locally constant cohomology of the manifold, which, as is well known, describes the category of flat bundles (see [6, I.2]).

A *Čech cocycle* of K with values in G , is a family $\xi := \{\xi_{aa'}\}$ of locally constant maps $\xi_{aa'} : V_{aa'} \rightarrow G$ satisfying the cocycle relations

$$\xi_{\tilde{a}\tilde{a}}(o)\xi_{\tilde{a}a}(o) = \xi_{\tilde{a}a}(o), \quad o \in V_{\tilde{a}\tilde{a}a}.$$

A cocycle ξ' is said to be *equivalent* to ξ if there is a family $u := \{u_a\}$ of locally constant maps $u_a : V_a \rightarrow G$ such that $\xi'_{\tilde{a}\tilde{a}}(o)u_{\tilde{a}}(o) = u_a(o)\xi_{\tilde{a}a}(o)$, $o \in V_{\tilde{a}\tilde{a}a}$. We denote the set of Čech cocycles and their equivalence classes by $Z_c^1(K, G)$ and $H_c^1(K, G)$, respectively. In the sequel, by a slight abuse of notation, we use the same symbols to denote cocycles and the corresponding equivalence classes.

Let us now consider the dual poset K° and its fundamental covering \mathcal{V}_0° . Note that $o \in V_a^\circ$ if and only if $a \leq^\circ o$, or equivalently if $o \leq a$. The aim is to establish a correspondence between $H^1(K, G)$, $H_c^1(K^\circ, G)$ and $H^1(K^\circ, G)$. Given $z \in Z^1(K, G)$, define

$$z_{\tilde{a}\tilde{a}}^c(o) := z(a; a, o) z(\tilde{a}; \tilde{a}, o)^{-1}, \quad o \in V_{\tilde{a}\tilde{a}}^\circ. \quad (22)$$

The definition is well posed since $a, \tilde{a} \leq^\circ o$ implies that $(a; a, o)$ and $(\tilde{a}; \tilde{a}, o)$ are 1-simplices of the nerve of K . Following the reasoning of [10, Lemma 4.10], z^c turns out to be the unique Čech cocycle of $Z_c^1(K^\circ, G)$ satisfying the equation $z_{\partial_1 b, \partial_0 b}^c(\partial_0 b) = z(\partial_1 b; \partial_1 b, \partial_0 b)$ for any 1-simplex of the nerve of K° . Thus we have an injective map

$$H^1(K, G) \rightarrow H_c^1(K^\circ, G), \quad z \mapsto z^c, \quad (23)$$

since any 1-cocycle is uniquely determined by its values on $\Sigma_1(K)$. By applying [10, Theorem 4.12], we also construct a bijective map

$$H_c^1(K^\circ, G) \rightarrow H^1(K^\circ, G), \quad \xi \mapsto \xi^{\mathbf{n}}, \quad (24)$$

where $\xi^{\mathbf{n}}(b) := \xi_{\partial_0 b, \partial_1 b}(|b|)$, $b \in \tilde{\Sigma}_1(K^\circ)$ and $\xi \in Z_c^1(K^\circ, G)$. If we compose (23) and (24), then we obtain an injective map

$$H^1(K, G) \rightarrow H^1(K^\circ, G), \quad z \mapsto z^\circ := (z^c)^{\mathbf{n}}. \quad (25)$$

We now observe that $z^{\circ\circ} = z$, $z \in H^1(K, G)$, implying that (25) and hence (23) are bijective. First note that

$$z^\circ(b) = z(\partial_0 b; \partial_0 b, |b|) z(\partial_1 b; \partial_1 b, |b|)^{-1}, \quad b \in \tilde{\Sigma}_1(K^\circ).$$

So, given $b \in \tilde{\Sigma}_1(K)$ we have

$$\begin{aligned} z^{\circ\circ}(b) &= z^\circ(\partial_0 b; \partial_0 b, |b|) z^\circ(\partial_1 b; \partial_1 b, |b|)^{-1} \\ &= z(\partial_0 b; \partial_0 b, \partial_0 b) z(|b|; |b|, \partial_0 b)^{-1} (z(\partial_1 b; \partial_1 b, \partial_1 b) z(|b|; |b|, \partial_1 b)^{-1})^{-1} \\ &= z(\sigma_0 \partial_0 b) z(|b|; |b|, \partial_0 b)^{-1} (z(\sigma_0 \partial_1 b) z(|b|; |b|, \partial_1 b)^{-1})^{-1} \\ &= z(|b|; \partial_0 b, |b|) z(|b|; |b|, \partial_1 b) \\ &= z(b), \end{aligned}$$

where we have applied the 1-cocycle identity to the last equality. The following result summarizes the previous considerations and incorporates an immediate consequence of Theorem 6.7.

Theorem 8.1. *The maps $z \mapsto z^c$, $\xi \mapsto \xi^{\mathbf{n}}$ induce one-to-one correspondences*

$$H^1(K, G) \rightarrow H_c^1(K^\circ, G) \quad , \quad H_c^1(K^\circ, G) \rightarrow H^1(K^\circ, G) \quad ,$$

defined so that, exchanging the role of K and K° , the diagram

$$\begin{array}{ccc} H^1(K, G) & \longrightarrow & H_c^1(K^\circ, G) \\ \uparrow & & \downarrow \\ H_c^1(K, G) & \longleftarrow & H^1(K^\circ, G) \end{array} \quad (26)$$

commutes. Moreover, the map (25) induces an isomorphism of categories from $\mathcal{P}(K, G)$ to $\mathcal{P}(K^\circ, G)$.

We now specialize taking our poset K to be a base for the topology of a manifold ordered under inclusion. Thus we can relate (principal) net bundles to well-known geometrical constructions, involving locally constant cohomology and flat bundles.

Let M be a paracompact, arcwise connected and locally contractible space. We denote by K the poset whose elements are the open, contractible subsets of M . A *locally constant cocycle* is given by a pair (A, f) , where $A \subseteq K$ is a locally finite open cover of M , and

$$f := \{f_{a'a} : a \cap a' \rightarrow G, a, a' \in A\}$$

is a family of locally constant maps satisfying the cocycle relations

$$f_{a''a'}(x)f_{a'a}(x) = f_{a''a}(x),$$

$x \in a \cap a' \cap a''$. A locally constant cocycle (\tilde{A}, \tilde{f}) is said to be *equivalent* to (A, f) if there are locally constant maps $v_{\tilde{a}a} : \tilde{a} \cap a \rightarrow G$, $a \in A$, $\tilde{a} \in \tilde{A}$, such that

$$v_{\tilde{a}a}(x)\tilde{f}_{\tilde{a}\tilde{a}'}(x) = f_{aa'}(x)v_{a'\tilde{a}'}(x),$$

where $x \in a \cap a' \cap \tilde{a} \cap \tilde{a}'$, $a, a' \in A$, $\tilde{a}, \tilde{a}' \in \tilde{A}$. Note that in locally constant cohomology we work with elements of A rather than with V_a , $a \in K$. This makes locally constant cocycles more manageable than cocycles defined over $\tilde{\Sigma}_0(K)$.

The set of locally constant cocycles is denoted by $Z_{lc}^1(M, G)$ and their equivalence classes by $H_{lc}^1(M, G)$ and called the (first) *locally constant cohomology* of M .

Now, if we endow G with the discrete topology, then $H_{lc}^1(M, G)$ coincides with the usual cohomology set classifying principal G -bundles over M (see Remark 8.3 below). In this way we obtain the following

Theorem 8.2. *Let M be a paracompact, arcwise connected, locally contractible space, and G a group. Then, there are isomorphisms*

$$H^1(K, G) \simeq \dot{\text{Hom}}(\pi_1(K), G) \simeq \dot{\text{Hom}}(\pi_1(M), G) \simeq H_{lc}^1(M, G),$$

where $\dot{\text{Hom}}(-, G)$ denotes the set of G -valued morphisms modulo inner automorphisms of G .

Proof. The first three isomorphisms have been already established [12] (see also [10]). The last isomorphism, instead, follows using the classical machinery developed in [13, I.13] for totally disconnected groups. In fact,

locally constant cocycles $(A, f) \in Z_{lc}^1(M, G)$ are in one-to-one correspondence with morphisms $\chi_{(A, f)}$ from $\pi_1(M)$ into G . A locally constant cocycle (\tilde{A}, \tilde{f}) is equivalent to (A, f) if and only if there is $g \in G$ such that $g \chi_{(A, f)} = \chi_{(\tilde{A}, \tilde{f})} g$. \square

Remark 8.3. Let G be a *topological* group. We consider the set $Z^1(M, G)$ of *cocycles* in the sense of [4, Chp.I], and the associated cohomology $H^1(M, G)$. We recall that elements of $Z^1(M, G)$ are pairs of the type (A, f) , where A is an open, locally finite cover of M , and f is a family of *continuous* maps $f_{aa'} : a \cap a' \rightarrow G$, $a, a' \in A$, satisfying the cocycle relations. It is well-known that such cocycles are in one-to-one correspondence with transition maps of principal bundles over M . The natural map

$$H_{lc}^1(M, G) \rightarrow H^1(M, G) \quad (27)$$

is *generally not injective*: in fact, inequivalent cocycles in $H_{lc}^1(M, G)$ may become equivalent in $H^1(M, G)$.

To make the isomorphisms of the previous theorem explicit, we provide an explicit map from $H_c^1(K^\circ, G) \simeq H^1(K, G)$ to $H_{lc}^1(M, G)$:

Proposition 8.4. *There is an isomorphism $H_c^1(K^\circ, G) \rightarrow H_{lc}^1(M, G)$.*

Proof. Let $A \subseteq K$ be a fixed locally finite, open cover, and ξ a Čech cocycle. Then, we have a family of locally constant maps

$$\xi_{aa'} : V_{aa'}^\circ \rightarrow G \quad , \quad a, a' \in \tilde{\Sigma}_0(K^\circ) \quad .$$

Now, since each $a \cap a'$ is an open set, we find $a \cap a' = \cup_{o \in V_{aa'}^\circ} o$. Since $\xi_{aa'}$ is locally constant on $V_{aa'}^\circ$, we can define the following locally constant function on $a' \cap a$:

$$\xi_{aa'}^{\text{lc}}(x) := \xi_{aa'}(o) \quad , \quad o \in V_{aa'}^\circ, x \in o \quad . \quad (28)$$

The family $\xi^{\text{lc}} := \{\xi_{aa'}^{\text{lc}}\}$ clearly satisfies the cocycle relations, so (A, ξ^{lc}) is a locally constant cocycle. If $\tilde{A} \subseteq K$ is another locally finite, open cover, then we obtain a locally constant cocycle $(\tilde{A}, \tilde{\xi}^{\text{lc}})$, with $\tilde{\xi}_{\tilde{a}\tilde{a}'}^{\text{lc}}(x)$, $x \in \tilde{a} \cap \tilde{a}'$, $\tilde{a}, \tilde{a}' \in \tilde{A}$, defined as in (28). By defining $v_{a\tilde{a}}(x) := \xi_{a\tilde{a}}(o)$, $o \in V_{a\tilde{a}}^\circ$, $x \in o$, we find $v_{a\tilde{a}}(x) \tilde{\xi}_{\tilde{a}\tilde{a}'}^{\text{lc}}(x) v_{a'\tilde{a}'}(x)^{-1} = \xi_{aa'}^{\text{lc}}(x)$, thus $(\tilde{A}, \tilde{\xi}^{\text{lc}})$ is equivalent to (A, ξ^{lc}) . This implies that the equivalence class of (A, ξ^{lc}) in $H_{lc}^1(M, G)$ does not depend on the choice of A . If ξ' is equivalent to ξ , then (A, ξ^{lc}) is equivalent to (A, ξ'^{lc}) for every open, locally finite, cover A ; thus, the map

$$\xi \mapsto (A, \xi^{\text{lc}}) \in H_{lc}^1(M, G) \quad , \quad \xi \in H_c^1(K^\circ, G) \quad , \quad (29)$$

is well defined at the level of cohomology classes. We prove the injectivity of (29). To this end, note that if (A, ξ^{lc}) and (A, ξ'^{lc}) are equivalent, then the above argument shows that $(\tilde{A}, \xi^{\text{lc}})$ and $(\tilde{A}, \xi'^{\text{lc}})$ are equivalent for every locally finite open cover $\tilde{A} \subseteq K$. Hence ξ and ξ' are equivalent. Finally, we prove the surjectivity of (29). If (A, f) is a locally constant cocycle, consider the associated representation $\chi : \pi_1(K) \rightarrow G$, and the corresponding Čech cocycle $\xi \in H_c^1(K, G)$, see Proposition 8.1, Theorem 8.2. In particular, the set $\xi_{aa'}$, $a, a' \in A$, is a locally constant cocycle (A, ξ) , equivalent to (A, f) by construction. \square

The above constructions can be summarized in the diagram

$$H_c^1(K^\circ, G) \xrightarrow{\simeq} H_{lc}^1(M, G) \longrightarrow H^1(M, G) \quad (30)$$

The first map (that emphasized by the symbol " \simeq ") is an isomorphism. The other is, in general, neither injective nor surjective. For example, when M is the 1-sphere S^1 and G the torus \mathbb{T} , we find $\mathbb{T} \simeq \text{Hom}(\mathbb{Z}, \mathbb{T}) \simeq H^1(K, \mathbb{T}) \simeq H_{lc}^1(S^1, \mathbb{T})$, whilst $H^1(S^1, \mathbb{T})$ is trivial.

9 Conclusions

The results of this paper demonstrate how the basic concepts and results of the theory of fibre bundles admit an analogue for net bundles over posets. In a sequel to this paper, we will show that this continues to be true for the K-theory of an Hermitian net bundle over a poset, even if the results diverge more in this case. In particular, we shall define the Chern classes for such bundles.

Our current research aims to develop a far-reaching but very natural generalization of the notion of net bundle. This involves replacing the poset, the base category of our net bundle, and the fibre category, here usually a group, by two arbitrary categories, a generalization leading us in the direction of fibred categories [3].

A Connection 1-cochains over posets

This appendix is intended to provide the reader with the notation and a very brief outline of the results of the cohomological description of connections over posets developed in [10].

The present paper treats principal bundles over posets having an arbitrary structure group. The lack of a differential structure forces us to use a cohomology taking values in the structure group G . Hence, in general, we will deal with a non-Abelian cohomology of posets. In [10] the coefficients for the non-Abelian n^{th} -degree cohomology of a poset K are an n -category associated with the group G . In the cited paper the set of n -cochains $C^n(K, G)$ is defined only for $n = 0, 1, 2, 3$. With this restriction, there is a coboundary operator $d : C^n(K, G) \rightarrow C^{n+1}(K, G)$ satisfying the equation $dd = \iota$ where ι denotes the *trivial* cochain. So the sets of n -cocycles and n -coboundaries are defined, respectively, by

$$Z^n(K, G) := C^n(K, G) \cap \text{Ker}(d), \quad B^n(K, G) := C^n(K, G) \cap \text{Im}(d).$$

Here we describe the category of 1-cochains in some detail, this being all we need.

For $n = 0, 1$, an n -cochain is just a mapping $v : \tilde{\Sigma}_n(K) \rightarrow G$. Given a 1-cochain $v \in C^1(K, G)$, we can and will extend v from 1-simplices to paths by defining for $p = b_n * \cdots * b_1$

$$v(p) := v(b_n) \cdots v(b_2) v(b_1).$$

Given $v, \tilde{v} \in C^1(K, G)$, a *morphism* f from \tilde{v} to v is a function $f : \tilde{\Sigma}_0(K) \rightarrow G$ satisfying the equation

$$f_{\partial_0 p} \tilde{v}(p) = v(p) f_{\partial_1 p},$$

for all paths p . We denote the set of morphisms from v_1 to v by (\tilde{v}, v) . There is an obvious composition law between morphisms given by pointwise multiplication making $C^1(K, G)$ into a category. The identity arrow $1_v \in (v, v)$ takes the constant value e , the identity of the group. Given a group homomorphism $\gamma : H \rightarrow G$ and a morphism $f \in (\tilde{v}, v)$ of 1-cochains with values in H then $\gamma \circ v$, defined by

$$(\gamma \circ v)(b) := \gamma(v(b)), \quad b \in \tilde{\Sigma}_1(K), \quad (31)$$

is a 1-cochain with values in G , and $\gamma \circ f$ defined by

$$(\gamma \circ f)_a := \gamma(f_a), \quad a \in \tilde{\Sigma}_0(K), \quad (32)$$

is a morphism of $(\gamma \circ \tilde{v}, \gamma \circ v)$. One checks at once that $\gamma \circ$ is a functor from $C^1(K, H)$ to $C^1(K, G)$, and that if γ is a group isomorphism, then $\gamma \circ$ is an isomorphism of categories.

Note that $f \in (v_1, v)$ implies $f^{-1} \in (v, v_1)$, where f^{-1} here denotes the composition of f with the inverse of G . We say that v_1 and v are *equivalent*,

written $v_1 \cong v$, whenever (v_1, v) is nonempty. Observe that a 1-cochain v is equivalent to the trivial 1-cochain ι if, and only if, it is a 1-coboundary. We will say that $v \in C^1(K, G)$ is *reducible* if there exists a proper subgroup $H \subset G$ and a 1-cochain $\tilde{v} \in C^1(K, H)$ with $\iota_{G,H} \circ \tilde{v}$ equivalent to v , where $\iota_{G,H}$ denotes the inclusion $H \subset G$. If v is not reducible it will be said to be *irreducible*.

A 1-cocycle of K with values in G is a 1-cochain z satisfying the equation

$$z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c), \quad c \in \tilde{\Sigma}_2(K). \quad (33)$$

The *category of 1-cocycles* with values in G , is the full subcategory of $C^1(K, G)$ whose set of objects is $Z^1(K, G)$. We denote this category by the same symbol $Z^1(K, G)$ as used to denote the corresponding set of objects. Clearly, 1-cohomology is strictly related to the first homotopy group. One first observes that any 1-cocycle z is *homotopic invariant*, i.e., $z(p) = z(q)$ whenever p and q are homotopic paths. Using this property,

$$Z^1(K, G) \cong H(\pi_1(K, a), G), \quad (34)$$

that is, the category $Z^1(K, G)$ is *equivalent* to the category $H(\pi_1(K, a), G)$ of group homomorphisms from $\pi_1(K, a)$ into G . Hence, if K is simply connected, then any 1-cocycle is a 1-coboundary. The set of *connections* with values in G is the subset $U^1(K, G)$ of those 1-cochains u of $C^1(K, G)$ satisfying the properties

$$\begin{aligned} (i) \quad & u(\bar{b}) = u(b)^{-1}, \quad b \in \tilde{\Sigma}_1(K), \\ (ii) \quad & u(\partial_0 c) u(\partial_2 c) = u(\partial_1 c), \quad c \in \Sigma_2(K). \end{aligned} \quad (35)$$

The *category of connection 1-cochains* with values in G , is the full subcategory of $C^1(K, G)$ whose set of objects is $U^1(K, G)$. It is denoted by $U^1(K, G)$ just as the corresponding set of objects.

The interpretation of 1-cocycles of $Z^1(K, G)$ as principal bundles over K with structure group G derives from the following facts. Any connection u of $U^1(K, G)$ induces a unique 1-cocycle $z \in Z^1(K, G)$ satisfying the equation

$$u(b) = z(b), \quad b \in \Sigma_1(K). \quad (36)$$

z is called the cocycle *induced* by u . Denoting the set of connections of inducing the 1-cocycle z by $U^1(K, z)$, we have that

$$U^1(K, G) = \dot{\cup} \{U^1(K, z) \mid z \in Z^1(K, G)\}, \quad (37)$$

where the symbol $\dot{\cup}$ means disjoint union. So, the set $U^1(K, z)$ can be seen as the set of connections of the principal bundle associated with z . We call the category of *connections inducing z* , the full subcategory of $U^1(K, G)$ whose objects belong to $U^1(K, z)$, and denote this category by $U^1(K, z)$ just as the corresponding set of objects.

The relation between cocycles (connections) taking values in different groups is easily established. Given a group homomorphism $\gamma : H \rightarrow G$, then the restriction of the functor $\gamma \circ$ to $Z^1(K, H)$ ($U^1(K, H)$) defines a functor from $Z^1(K, H)$ into $Z^1(K, G)$ and ($U^1(K, H)$ into $U^1(K, G)$). This functor is an isomorphism when γ is a group isomorphism.

References

- [1] R. Bott, L.W. Tu. *Differential forms in Algebraic Topology*. 1982 Springer-Verlag New York, Inc.
- [2] P. Gabriel, M. Zisman. *Calculus of fractions and homotopy theory* Springer-Verlag Berlin, Heidelberg, 1967.
- [3] A. Grothendieck. *Catégories fibrées et descente*”, Schémas en groupes (Sém. Géométrie Algébrique, Inst. Hautes Études Sci., 1960/61), Fasc. 2, Exposé 6, Inst. Hautes Études Sci., Paris, 1961.
- [4] M. Karoubi. *K-theory* Springer-Verlag Berlin, Heidelberg, New York, 1978.
- [5] S. Kobayashi, K. Nomizu. *Foundations of differential geometry*. Vol.I Wiley, New York 1963.
- [6] S. Kobayashi. *Differential geometry of complex vector bundles* Publications of the Mathematical Society of Japan, Vol. 15, Princeton University Press, Princeton, 1987.
- [7] J.P. May. *Simplicial objects in algebraic topology*. D. Van Nostrand Co., Inc., Princeton, N.J (1967).
- [8] J.E. Roberts. *Mathematical aspects of local cohomology*. In: *Algèbres d’opérateurs et leurs applications en physique mathématique*. (Proc. Colloq. Marseille, 1977) 321–332. Colloq. Internat. CNRS, **274**, CNRS, Paris (1979).

- [9] J.E. Roberts.: More lectures in algebraic quantum field theory.
In: S. Doplicher, R. Longo (eds.) *Noncommutative geometry*
C.I.M.E. Lectures, Martina Franca, Italy, 2000. Springer 2003.
- [10] J.E. Roberts, G. Ruzzi. *A cohomological description of connections and curvature over posets.*, Theory and Applications of Categories, **16**, no.30, 855–895, (2006).
- [11] J.E. Roberts, G. Ruzzi, E. Vasselli. In preparation.
- [12] G. Ruzzi. *Homotopy of posets, net-cohomology and superselection sectors in globally hyperbolic spacetimes.*, Rev. Math. Phys. **17**, no. 9, 1021–1070, (2005).
- [13] N. Steenrod. *The topology of fibre bundles*. Princeton University Press, Princeton NJ, 1951.